Functional Analysis (WS 19/20), Big Homework 2

deadline: 14/11/2019 (both groups, 13:45, room 3140 - after class)

Important: Each problem should be solved on a separate piece of paper signed with your name, student id number and group number (1 or 2).

1. Let $(E, \|\cdot\|_E)$ be a normed space and $f : [0,1] \to E$ be a continuous map. Prove that

$$\|f\|_\infty = \sup\{\|f(x)\|_E : x \in [0,1]\}$$

defines a norm on the space $C([0,1]; E)$, i.e. space of continuous E-valued functions.

Moreover, suppose additionally that $(E, \|\cdot\|_E)$ is a Banach space. Prove that $C([0,1]; E)$ is also a Banach space.

2. Let $(E, \|\cdot\|_E)$ be a Banach space and $A : E \to E$ a bounded linear operator. Suppose that there is a natural number $n \in \mathbb{N}$ and real numbers $c_1, ..., c_n$ such that

$$I + c_1 A + ... + c_n A^n = 0,$$

where I is the identity operator. Prove that A^{-1} exists and it is a bounded linear operator.

3. Consider

$$X = \{f \in L^2(-1,1) : f(x) = f(-x)\}$$

as a subspace of $L^2(-1,1)$. Find explicitly $X^\perp$ in $L^2(-1,1)$ and compute explicitly projection operator on the space X.

4. Let $(E, \|\cdot\|_E)$ be a Banach space and $\varphi : E \to \mathbb{R}$ be a linear functional on E.

   (a) Prove that if $\varphi \neq 0$, then there is a one dimensional subspace $F \subset E$ such that

   $$E = \ker\varphi \oplus F$$

   i.e. for all $x \in E$, there are uniquely determined $y \in \ker\varphi$ and $z \in F$ such that $x = y + z$.

   (b) Prove that $\varphi \in E^*$ (i.e. it is bounded) if and only if its kernel is closed in E.

   Recall: By a kernel of a linear functional φ we mean the set $\ker\varphi = \{x \in E : \varphi(x) = 0\}$.