Functional Analysis (WS 19/20), Big Homework 3

deadline: 5/12/2019 (group 1), TBD (group 2)

Important: Each problem should be solved on a separate piece of paper signed with your name, student id number and group number (1 or 2).

1. (**Clarkson's first inequality**) Let $2 \leq p < \infty$ and $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Prove that for all $f, g \in L^p(\Omega, \mathcal{F}, \mu)$:
$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{\|f\|_p^p + \|g\|_p^p}{2}.$$
Conclude that $L^p(\Omega, \mathcal{F}, \mu)$ is uniformly convex for $2 \leq p < \infty$ (see Problem P5 in Problem Set 5 for definition and uniform convexity of Hilbert spaces). *Hint:* Simplify the problem to some inequality for real numbers. *Remark:* Compare with Problem P5 in Problem Set 5.

2. Prove that the map $T : l^1 \to (c_0)^*$ given with
$$(Ty)(x) = \sum_{i=1}^{\infty} x_i y_i$$
is well-defined, injective, surjective and isometry (i.e. $\|y\|_{l_1} = \|Ty\|_{(c_0)^*}$). Conclude that $(c_0)^* = l_1$. *Hint*: Recall Schauder basis for c_0: if $x = (x_1, x_2, ...) \in c_0$ then projection of x given with $x^k = (x_1, ..., x_k, 0, 0, ...)$ converges to x in c_0.

3. Let $(E, \|\cdot\|)$ be a Banach space and $A \subset E$ be its subset. Suppose that for every $f \in E^*$, the set
$$f(A) = \{f(x) : x \in A\}$$
is bounded in $\mathbb{R}$. Prove that A is a bounded set in E (i.e. one can find a ball $B(0, R)$ for some $R > 0$ such that $A \subset B(0, R)$).

4. Let X be a vector space (not necessarily normed or Banach) over $\mathbb{R}$. Let φ, φ_1, ..., φ_k be linear functionals on $\mathbb{R}$ (i.e. linear maps from X to $\mathbb{R}$). Suppose that
$$(\forall_{i=1,...,k}\ \varphi_i(v) = 0) \implies \varphi(v) = 0.$$
Prove that φ is a linear combination of $\varphi_1, ..., \varphi_k$, i.e. there are real numbers $\lambda_1, ..., \lambda_k$ such that $\varphi = \sum_{n=1}^{k} \lambda_n \varphi_n$. *Hint:* Consider $F(x) = (\varphi_1(x), ..., \varphi_k(x), \varphi(x))$.

5. Let $1 < p < \infty$.
   - Let $\alpha, \beta \in \mathbb{R}$ and set $f(x) = \begin{cases} \alpha & x \in [0, 1/2] \\ \beta & x \in (1/2, 1] \end{cases}$. Then, extend f periodically on $\mathbb{R}$. Finally, we set $f_n(x) = f(nx)$. Decide whether sequence $\{f_n\}_{n \in \mathbb{N}}$ converges weakly in $L^p(0, 1)$ and if yes, determine the limit.
   - Suppose $u \in L^p(\mathbb{R})$. We set $g_n(x) = u(x + n)$. Decide whether sequence $\{g_n\}_{n \in \mathbb{N}}$ converges weakly in $L^p(\mathbb{R})$ and if yes, determine the limit.
   - Let $h_n(x) = n^{1/p} e^{-nx}$. Prove that $g_n \to 0$ a.e., sequence $\{g_n\}_{n \in \mathbb{N}}$ is bounded in $L^p(0, 1)$, g_n converges weakly[1] to 0 in $L^p(0, 1)$ but does not converge in norm (strongly) in $L^p(0, 1)$.

[1] Update on 29/11/2019: converges weakly to 0 in $L^p(0, 1)$ but does not strongly (in norm).