Functional Analysis (WS 19/20), Big Homework 4

deadline: 19/12/2019 (group 1), TBD (group 2)

Important: Each problem should be solved on a separate piece of paper signed with your name, student id number and group number (1 or 2).

1. In the following we will construct conditional expectation using Radon-Nikodym theorem:

   (a) Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space where μ is nonnegative and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Let $f \in L^1(\mu)$ and ν be a restriction of μ to $\mathcal{G}$. Prove that there exists $g \in L^1(\nu)$ (in particular: g is $\mathcal{G}$-measurable) such that

   $$\int_E f \, d\mu = \int_E g \, d\nu \qquad \forall_{E \in \mathcal{G}}.$$

   Moreover, justify that g is uniquely determined up to the null sets of ν.[1]

   (b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a real-valued random variable such that $\mathbb{E}|X| < \infty$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Prove that there exists a $\mathcal{G}$-measurable random variable Y such that

   $$\mathbb{E} X \mathbb{1}_A = \mathbb{E} Y \mathbb{1}_A \qquad \forall_{A \in \mathcal{G}}.$$

   We usually write $Y = \mathbb{E}(X|\mathcal{G})$ and call Y a conditional expectation of X with respect to $\mathcal{G}$.

2. Let E be a normed space and $f : E \to \mathbb{R}$ be a convex and lower semicontinuous function on E, i.e. for any sequence $\{x_n\} \subset E$: $x_n \to x$ in E implies $f(x) \leq \liminf_{n \to \infty} f(x_n)$.

   (a) Prove that the epigraph of f, i.e. $\mathrm{epi}(f) = \{(x, \lambda) \in E \times \mathbb{R} : \lambda \geq f(x)\}$, is a convex and closed set in $E \times \mathbb{R}$.

   (b) Prove that there is a family of affine functions

   $$\mathcal{A} \subset \{\varphi(x) + b : \varphi \in E^*, b \in \mathbb{R}\}$$

   such that $f(x) = \sup_{\phi \in \mathcal{A}} \phi(x)$. *Remark*: This is a generalization of standard fact that a convex function is a supremum of the family of affine supporting functions. *Hint*: Hahn-Banach.

   (c) Conclude that there are constants $a, b \in \mathbb{R}$ such that $f(x) \geq a + b\|x\|$.

3. Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined with $Tf(x) = \mathrm{sgn}(x) f(x+1)$ on the complex Hilbert space $L^2(\mathbb{R})$. Prove that T is a well - defined bounded linear operator and compute T^*.

4. Consider the right and left shifts operators on the complex Hilbert space $l^2(\mathbb{N})$ (we usually denote this space with l^2) defined with

   $$Rx = (0, x_1, x_2, ...), \qquad Lx = (x_2, x_3, x_4, ...).$$

   Find point, continuous and residual parts of spectrum of R and L.[2]

[1] This is often used in the following setting: one works with integral $\int_E f \, d\mu$ but it is useful to replace it with an integral $\int_E g d\mu$ where g is measurable with respect to some smaller σ-algebra that is usually generated by some given sets. This is, for instance, a technical point in the proof of celebrated Dunford-Pettis Theorem asserting that bounded sequences in L^1 are weakly compact if and only if they are uniformly integrable. Unlike L^p with $1 < p < \infty$, it is not true that bounded sequence in L^1 has a converging subsequence in a weak sense - it is easy to construct an example. See Theorem 1.38 in L. Ambrosio, N. Fusco, D. Pallara *Functions of Bounded Variation and Free Discontinuity Problems*.

[2] *Hint:* One can make this problem easier by finding some adjoint relationship between L and R, see Problem R8 in PS8.