Functional Analysis (WS 19/20), Big Homework 5

deadline for group 1: 16.01.2020 (problem 1, 2), 23.01.2020 (problem 3, 4)

Important: Each problem should be solved on a separate piece of paper signed with your name, student id number and group number (1 or 2).

1. Let $A : H \to H$ be self-adjoint and compact linear operator on a separable Hilbert space H. Let $n \in \mathbb{N}$. Prove that there exists a bounded linear operator $B : H \to H$ such that $B^n = A$. Is this operator uniquely determined?

2. (**Young's inequality**) Prove Young's convolutional inequality: if $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ then $f * g \in L^r(\mathbb{R}^d)$ where $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, 1 \leq p, q, r \leq \infty$. Moreover,

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

   Do not use Riesz–Thorin interpolation in this Problem!

3. Let E, F be Banach spaces and $K : E \to F$ a compact linear operator. Prove that for every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging weakly (i.e. $x_n \rightharpoonup x$) we have $Kx_n \to Kx$. Hence, compact operators map weakly converging subsequences to the strongly converging ones.

   Hint: Prove that in the normed space Y, the sequence $\{y_n\}_{n\in\mathbb{N}} \subset Y$ converges to $y \in Y$ if and only if every subsequence of $\{y_n\}_{n\in\mathbb{N}}$ has a further subsequence converging to y.

4. (**Dini's Theorem**) Fix $x \in \mathbb{R}$. Let f be a continuous function that is 1-periodic, i.e. $f(y+1) = f(y)$ for all $y \in \mathbb{R}$. Suppose that f satisfies for some $\delta > 0$ the following condition:

$$\int_{|t|<\delta} \frac{|f(x+t) - f(x)|}{|t|}\, dt < \infty.$$

   Use properties of the Dirichlet kernel and the proof of Riemann Localization Principle to prove that the Fourier series of f converges at x to $f(x)$.

   Remark: In particular, if f is a globally Lipschitz function, i.e. there is $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}$, then Fourier series of f converges to f.