Solution to the W7 Problem from PS 6

($\Leftarrow$): Let $x_n \to x$, where $x_n \in C$. By W4 we see that $x_n \rightharpoonup x$ and since C is closed for weak convergence, then $x \in C$, so C is closed for convergence in norm.

($\Rightarrow$): Let $x_n \rightharpoonup x$, where $x_n \in C$, and assume $x \notin C$. By Hahn-Banach theorem (v.2), as C and $\{x\}$ are convex, C is closed and $\{x\}$ is compact, then there exists $\varphi \in X^*$ and $\lambda \in \mathbb{R}$ such that

$$\varphi(c) < \lambda < \varphi(x)$$

for all $c \in C$. Therefore $\limsup_{n\to\infty} \varphi(x_n) \leqslant \lambda < \varphi(x)$, meaning that $\varphi(x_n) \not\to \varphi(x)$. This contradicts $x_n \rightharpoonup x$, so $x \in C$, as desired.