Homework for 30/10/2019

Let E be Banach space, $T: E \to E^*$ s.t. $(Tx)(x) \geq 0$ $\forall_{x \in E}$.
Prove that T is bounded.

Solution: We wish to apply Closed Graph Theorem.

$$G(T) = \{ (x, Tx) \in E \times E^* : \ x \in E \}.$$

Let $(x_n, Tx_n)_{n \geq 1} \subset G(T)$ such that $\begin{cases} x_n \to x & \text{in } E \\ Tx_n \to y & \text{in } E^* \end{cases}$.

We want to prove that $y = Tx$. To this end, fix $z \in E$ and consider sequence $z_\varepsilon = (x - x_n) + \varepsilon z$. Then

$$0 \leq (Tz_\varepsilon)(z_\varepsilon) = (T(x-x_n))(x-x_n) + \varepsilon (T(x-x_n)) z$$
$$+ \varepsilon T(z)(x-x_n) + \varepsilon^2 T(z)(z).$$

Term $(T(x-x_n))(x-x_n)$ converges to 0 as $n \to \infty$ (by triangle inequality)
Similarly $\varepsilon T(z)(x-x_n)$ converges to 0 as $n \to \infty$ since $Tz \in E^*$.
Term $\varepsilon (T(x-x_n))(z)$ converges to $\varepsilon (Tx - y)(z)$. We have:

$$0 \leq \varepsilon^2 (Tz)(z) + \varepsilon (Tx - y)(z) \qquad / : \varepsilon$$

and let $\varepsilon \to 0$ to deduce $\quad 0 \leq (Tx - y)(z) \quad \forall_{z \in E}$.
As this holds for all $z \in E$, we take $-z$ and have $0 = (Tx - y)(z)$
so $\underline{Tx = y}$ in E^*.

$\square$.