Techniques to study spectrum of the operator

1) study point spectrum: this is always a simple computation

2) study for which $\lambda \in \mathbb{C}$ operator $A-\lambda I$ can be explicitly inverted [example: S7 in PS8 with $Mf(x) = xf(x)$ on $L^2_{[0,1]}$, for $\lambda \cancel{\in} [0,1)$ we get $(M-\lambda I)^{-1}f(x) = \frac{1}{\lambda - x} f(x). \in L^2$].

3) study norm of the operator: this gives you that $|\sigma(A)| \leq \|A\|$.

4) use that spectrum is compact so closed: if you know that $\{|\lambda| < 1\} \subset \sigma(A)$ then $\{|\lambda| \leq 1\} \subset \sigma(A)$.

5) $A - \lambda I$ cannot be invertible if $\exists_{\{x_n\} \subseteq H,\ \varepsilon_n \subset \mathbb{R}}$ such that $\|(A-\lambda I)x_n\| \leq \varepsilon_n \|x_n\|$ and $\varepsilon_n \to 0$ (see S15 and S13 in PS8)

6) adjoint techniques are in particular useful when one has to classify elements of spectrum [point / continuous / residual].
Note that $\ker(A)^{\perp} = \overline{\mathrm{im}(A^*)}$ and $\mathrm{im}(A)^{\perp} = \ker(A^*)$.
Similar relations can be deduced by $A^{**} = A$. Also note that $(A-\lambda I)^* = A^* - \overline{\lambda} I$ note the conjugate here!
(see Problem 4 in BH4)

7) if operator is self-adjoint (i.e. $S^* = S$) then S has real spectrum.

8) If operator is compact, then 0 is always in its spectrum. Moreover, any other element is in its point spectrum (so there is 0 and eigenvalues). Moreover, 0 can be the only limit point in the spectrum.
This can be also used to prove that <u>operator is not compact</u>