Problem from 2nd midterm:

Let $s \in \mathbb{R}$. Consider $T_s : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ (spaces on $\mathbb{C}$) def. as $T_s f(x) = f(x-s)$. Find $\sigma(T_s)$.

Hint $T_s^* = \underbrace{T_{-s}}_{=(T_s)^{-1}}$. (part (A) of the problem)

SOLUTION: Clearly $\|T_s\| = 1$ for all $s \in \mathbb{R}$. We study operator $T_s - \lambda I$ for $s \neq 0$, $\lambda \in \mathbb{C}$.

Recall that $I - A$ is invertible if $\|A\| < 1$. Hence

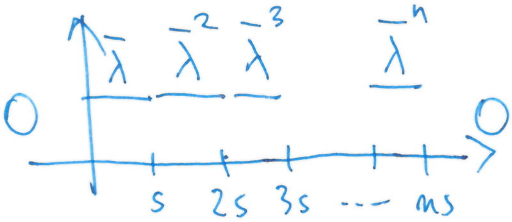
- $T_s - \lambda I = -\lambda (I - T_s/\lambda)$ is invertible if $|\lambda| > 1$.
- $T_s - \lambda I = T_s(I - \lambda T_s^{-1}) = T_s(I - \lambda T_{-s})$ is invertible if $|\lambda| < 1$

This shows that $\sigma(T_s) \subset \{|\lambda| = 1\}$. We claim that in fact

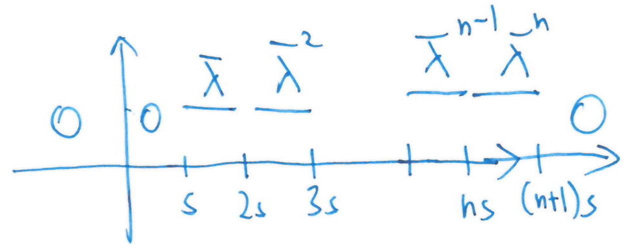
$$\boxed{\sigma(T_s) = \{|\lambda| = 1\}} \qquad (s \neq 0).$$

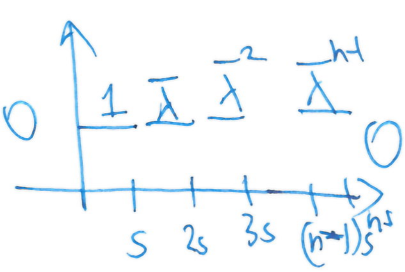
We recall the following fact: if $\exists_{x_n} \exists_{\varepsilon_n \to 0} \; \|Ax_n\| \leq \varepsilon_n \|x_n\|$ then A cannot be invertible.

Let $\lambda \in \{|\lambda| = 1\}$. We define f_n as follows:

$f_n :=$ 

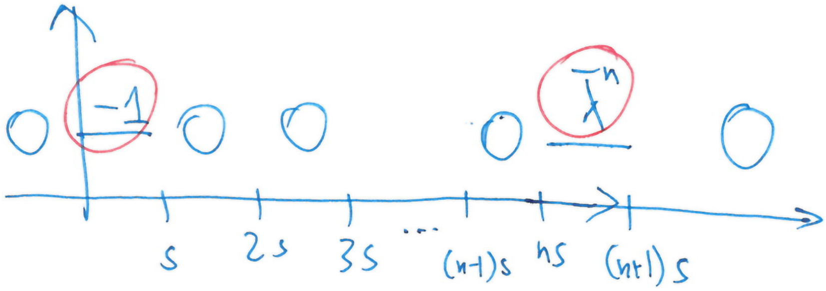
$f_n \in L^2(\mathbb{R})$, $\|f_n\|_{L^2} = \sqrt{n}$.

$T_s f_n :=$ 

$\lambda f_n :$ 

So we conclude:

$(T_s - \lambda I) f_n =$



$$\Rightarrow \|(T_s - \lambda I) f_n\|_{L^2(\mathbb{R})} = \sqrt{2}.$$

$$\Rightarrow \|(T_s - \lambda I) f_n\|_{L^2(\mathbb{R})} = \sqrt{\frac{2}{n}} \, \|f_n\|_{L^2(\mathbb{R})} \leq \sqrt{\frac{2}{n}} \, \|f_n\|_{L^2(\mathbb{R})}$$

$\Rightarrow$ $(T_s - \lambda I)$ cannot be invertible.

Therefore $\sigma(T_s) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, $s \neq 0$.

Finally, we consider case $s = 0$, so T_s becomes identity operator on $L^2(\mathbb{R})$. Then,

$$T_s - \lambda I = I - \lambda I = (1-\lambda) I \quad \text{so} \quad \sigma(T_s) = \{1\} \; (s=0).$$

$\square$.