Functional Analysis (WS 19/20), Problem Set 3

(Baire Category Theorem and Uniform Boundedness Principle)

Baire Category Theorem

B1. Let $(X, \|\cdot\|_X)$ be an infinite dimensional Banach space. Prove that X has uncountable Hamel basis. *Note*: This is Problem A2 from Problem Set 1.

B2. Consider subset of bounded sequences

$$A = \{x \in l^\infty : \text{ only finitely many } x_k \text{ are nonzero}\}.$$

Can one define a norm on A so that it becomes a Banach space? Consider the same question with the set of polynomials defined on interval $[0, 1]$.

B3. Prove that the set $L^2(0, 1)$ has empty interior as the subset of Banach space $L^1(0, 1)$.

B4. Let $f : [0, \infty) \to [0, \infty)$ be a continuous function such that for every $x \in [0, \infty)$, $f(kx) \to 0$ as $k \to \infty$. Prove that $f(x) \to 0$ as $x \to \infty$.

B5. (**Uniform Boundedness Principle**) Let $(X, \|\cdot\|_X)$ be a Banach space and $(Y, \|\cdot\|_Y)$ be a normed space. Let $\{T_\alpha\}_{\alpha \in A}$ be a family of bounded linear operators between X and Y. Suppose that for any $x \in X$,

$$\sup_{\alpha \in A} \|T_\alpha x\|_Y < \infty.$$

Prove that $\sup_{\alpha \in A} \|T_\alpha\| < \infty$.

Uniform Boundedness Principle

U1. Let F be a normed space $C[0, 1]$ with $L^2(0, 1)$ norm. Check that the formula

$$\varphi_n(f) = n \int_0^{\frac{1}{n}} f(t)\, dt$$

defines a bounded linear functional on F. Verify that for every $f \in F$, $\sup_{n \in \mathbb{N}} |\varphi_n(f)| < \infty$ but $\sup_{n \in \mathbb{N}} \|\varphi_n\| = \infty$. Why Uniform Boundedness Principle is not satisfied in this case?

U2. (**pointwise convergence of operators**) Let $(X, \|\cdot\|_X)$ be a Banach space and $(Y, \|\cdot\|_Y)$ be a normed space. Let $\{T_n\}_{n \in \mathbb{N}}$ be a family of bounded linear operators between X and Y such that for every $x \in X$, the sequence $T_n x$ converges to a limit denoted by Tx. Prove that

(a) $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$,

(b) T defines a bounded linear operator,

(c) $\|T\| \le \liminf_{n \to \infty} \|T_n\|$,

(d) if $x_n \to x$ in X then $T_n x_n \to Tx$ in Y.

U3. Give an example demonstrating that under assumptions of Problem U2., one cannot hope that $T_n \to T$ strongly (i.e. in operator norm) even if $(Y, \|\cdot\|)$ is also a Banach space.

U4. Let $(X, \|\cdot\|_X)$ be a Banach space and $A \subset X^*$ such that for every $x \in X$ the set

$$\{\varphi(x) : \varphi \in A\}$$

is bounded in $\mathbb{R}$. Prove that A is a bounded subset of X^*, i.e. $\sup\{\|\varphi\| : \varphi \in A\} < \infty$.

U5. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces. Let $a : E \times F \to \mathbb{R}$ be a bilinear form such that

- for fixed $x \in E$, the map $F \ni y \mapsto a(x, y)$ is continuous (so it belongs to F^*),
- for fixed $y \in F$, the map $E \ni x \mapsto a(x, y)$ is continuous (so it belongs to E^*).

Prove that there exists a constant C such that

$$|a(x, y)| \leq C\|x\|_E\|y\|_F$$

for all $x \in E$ and $y \in F$. Thus, linear maps that are separately continuous are actually jointly continuous. *Hint*: Problem U4. may be useful.

U6. Let X be the space of polynomials in one variable defined on $(0, 1)$ equipped with the $L^1(0, 1)$ norm. We define a bilinear map: for $f, g \in X$, we put

$$\mathcal{B}(f, g) = \int_0^1 f(t)\, g(t)\, dt.$$

Check that $\mathcal{B}$ is separately continuous but it is not jointly continuous (in the sense of Problem U5.).

U7. Let $(x_n)_{n\geq 1}$ be a sequence of real numbers such that whenever $(y_n)_{n\geq 1}$ is a real sequence converging to 0 we have that $\sum_{n\geq 1} x_n y_n$ is convergent. Prove that $\sum_{n\geq 1} |x_n|$ is convergent. *Hint*: for $y \in c_0$, consider $T_n \in (c_0)^*$ defined with $T_n(y) = \sum_{k=1}^{n} x_k y_k$.