Functional Analysis (WS 19/20), Problem Set 8

(spectrum and adjoints on Hilbert spaces)[1]

In what follows, let H be a **complex** Hilbert space.

Let $T : H \to H$ be a bounded linear operator. We write $T^* : H \to H$ for **adjoint** of T defined with

$$\langle Tx, y \rangle = \langle x, T^* y \rangle.$$

This operator exists and is uniquely determined by Riesz Representation Theorem.

Spectrum of T is the set $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not have a bounded inverse}\}$.
Resolvent of T is the set $\rho(T) = \mathbb{C} \setminus \sigma(T)$.

Basic facts on adjoint operators

R1. ♣ Adjoint T^* exists and is uniquely determined.

R2. ♣ Adjoint T^* is a bounded linear operator and $\|T^*\| = \|T\|$. Moreover, $\|T^*T\| = \|T\|^2$.

R3. ♣ Taking adjoints is an involution: $(T^*)^* = T$.

R4. ♣ Adjoints commute with the sum: $(T_1 + T_2)^* = T_1^* + T_2^*$.

R5. ♣ For $\lambda \in \mathbb{C}$ we have $(\lambda T)^* = \overline{\lambda} T^*$.

R6. ♣ Let T be a bounded invertible operator. Then, $(T^*)^{-1} = (T^{-1})^*$.

R7. ♣ Let T_1, T_2 be bounded operators. Then, $(T_1 T_2)^* = T_2^* T_1^*$.

R8. ♣ We have relationship between kernel and image of T and T^*:

$$\ker T^* = (\text{im } T)^{\perp}, \qquad (\ker T^*)^{\perp} = \overline{\text{im } T}$$

It will be helpful to prove that if $M \subset H$ is a linear subspace, then $\overline{M} = \left(M^{\perp}\right)^{\perp}$. Btw, this covers all previous results like if N is a finite dimensional linear subspace then $N = \left(N^{\perp}\right)^{\perp}$ (because N is closed).

Computation of adjoints

M1. ☺Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a complex matrix. Find A^*.

M2. ♣ ☺Let $H = l^2(\mathbb{Z})$. For $x = (..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...) \in H$ we define the right shift operator with $(Rx)_k = x_{k-1}$. Find $\|R\|$, R^{-1} and R^*. Similarly, one can consider the left shift operator L.

M3. ☺Let $K : L^2(0,1) \to L^2(0,1)$ be defined with $Kf(x) = \int_0^x f(y)$. Prove that K is a bounded linear operator and compute K^*.

[1] A useful reference for this topic is Chapter 9 of the book *Applied Analysis* by John Hunter and Bruno Nachtergaele available online at https://www.math.ucdavis.edu/ hunter/book/pdfbook.html. It may be helpful to read Wikipedia articles: "Hermitian adjoint", "Spectrum (functional analysis)" and "Decomposition of spectrum (functional analysis)".

M4. ♣ ☺Let $M \subset H$ be a closed subspace and P_M be an orthogonal projection on M. Find $(P_M)^*$.

M5. ☺Let $A : H \to H$ be a bounded operator. Recall that e^A exists as a series $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ converging in the operator norm. Compute $\left(e^A\right)^*$.

M6. ☺Let $T : L^2(0,1) \to L^2(0,1)$ be defined with

$$Tf(x) = \int_0^1 k(x,y)f(y)dy$$

for some bounded and measurable function $k(x,y)$. Find the adjoint of T. *Remark:* This operator is called Hilbert-Schmidt operator.

M7. ☺Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined with $Tf(x) = \operatorname{sgn}(x)f(x+1)$. Prove that T is well - defined and find T^*.

Spectrum of an operator on Hilbert space

S1. ♣ Let A be a bounded operator. Prove that $\sigma(A)$ can be decomposed into three disjoint parts:

(a) point spectrum: $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not injective,

(b) continuous spectrum: $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is injective but not surjective and image of $A - \lambda I$ is dense in H,

(c) residual spectrum: $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is injective but not surjective and image of $A - \lambda I$ is not dense in H

If $\lambda \in \mathbb{C}$ belongs to the point spectrum, we say that λ is an eigenvalue of A.

S2. ♣ Prove that $\sigma(A) \subset B(0, \|A\|) \subset \mathbb{C}$.

S3. ♣ Prove that $\rho(A)$ is an open subset of $\mathbb{C}$. Conclude that $\sigma(A)$ is a compact subset of $\mathbb{C}$. Compare with Problem S11.

S4. ★ Prove that $\sigma(A)$ cannot be empty. *Hint*: Liouville theorem applied to the function $\mathbb{C} \ni \lambda \mapsto (A - \lambda I)^{-1}$. Is it the same for bounded operators on real Hilbert spaces?[2]

S5. Let p be a polynomial. Prove that if $\sigma(p(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}$.

S6. ☺Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a complex matrix. Prove that A has a purely point spectrum.

S7. ☺Let $M : L^2(0,1) \to L^2(0,1)$ be defined with $(Mf)(x) = xf(x)$. Find point, continuous and residual parts of the spectrum of M. *Remark:* The result is that M has purely continuous spectrum.

S8. ☺Let $A : l^2 \to l^2$ be defined with $Ax = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$. Find point, continuous and residual parts of the spectrum of A. *Remark:* The result is that A has purely residual spectrum.

S9. ♣ Let A be a bounded operator. We say that $\lambda \in \mathbb{C}$ belongs to an *approximate spectrum* of A if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\|x_n\| = 1$ and $(A - \lambda I)x_n \to 0$. Prove that if λ is in approximate spectrum of A, then $\lambda \in \sigma(A)$. Moreover, prove that approximate spectrum contains point and continuous parts of spectrum.

[2] It seems that on complex Hilbert spaces everything is more complex...

S10. Find an example of operator A on Hilbert space H such that its residual and approximate parts of spectrum are not empty and:

(a) its residual spectrum is not disjoint with approximate spectrum,

(b) its residual spectrum is disjoint with approximate spectrum.

In case this is impossible, prove that there is no such operator.

S11. ♣ Let $K \subset \mathbb{C}$ be a nonempty and compact subset. Prove that there is an operator T on $L^2(0,1)$, such that $\sigma(T) = K$. *Remark*: $L^2(0,1)$ can be replaced here with any separable Hilbert space.

S12. ☺Let G be a multiplication operator on $L^2(\mathbb{R})$ defined with $(Gf)(x) = g(x)f(x)$ for some bounded and continuous function g. Prove that

$$\sigma(G) = \overline{\{g(x) : x \in \mathbb{R}\}}$$

where upper line denotes the closure of the set. Can operator G have eigenvalues?

S13. ☺Consider the right shift operator R on $l^2(\mathbb{Z})$. Prove that:

(a) The point spectrum of R is empty.

(b) The image of $R - \lambda I$ is $l^2(\mathbb{Z})$ for $\lambda \in \mathbb{C}$ such that $|\lambda| \neq 1$.

(c) The spectrum of S is purely continuous and contains only the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

S14. ☺Consider the right and left shifts operators on $l^2(\mathbb{N})$ (we usually denote this space with l^2) defined with

$$Rx = (0, x_1, x_2, ...), \qquad Lx = (x_2, x_3, x_4, ...).$$

Find point, continuous and residual parts of spectrum of R and L. *Remark:* It is rather clear that the result is different for R and L.

S15. Let A be a bounded operator on Hilbert space H. Suppose there is a sequence $\{x_n\} \subset H$ and $\{\epsilon_n\} \subset \mathbb{R}$ such that $\epsilon_n \to 0$ and

$$\|Ax_n\| \leq \epsilon_n \|x_n\|.$$

Prove that A does not have a bounded inverse. In particular, it does not have an inverse as bounded linear isomorphisms have bounded inverses.

S16. Let M be a multiplication operator from Problem S7. Find the spectrum of the operator

$$M^2 + M - 2.$$

Additional problems from the lecture

A1. Let $M \subset H$ be a linear subspace such that $M^\perp = \{0\}$. Prove that M is dense in H.

A2. Show that assertion of Problem A1. is not valid if M is assumed to be just a subset of H.