Problem Set 9: Spectrum of SA and Compact operators

(S1) Let $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$, Im = imaginary part.

- First, $A-\lambda I$ is injective. Indeed,

$$\operatorname{Im}\left[\lambda \|x\|^2\right] = \operatorname{Im}\left[\langle(\lambda I - A)x, x\rangle\right] \quad \text{as} \quad \langle Ax, x\rangle \in \mathbb{R}$$

(because $\overline{\langle Ax, x\rangle} = \langle x, Ax\rangle \overset{A=A^*}{=} \langle Ax, x\rangle$)

Therefore, $|\operatorname{Im}\lambda| \cdot \|x\|^2 \leq \|(A-\lambda I)x\| \, \|x\|$ by Cauchy-Schwartz so

$$\|x\| \leq \frac{1}{|\operatorname{Im}\lambda|} \|(A-\lambda I)x\| \quad (*) \Rightarrow \underline{A-\lambda I \text{ is injective for all } \lambda \text{ with } \operatorname{Im}\lambda \neq 0}$$

- Second, image of $(A-\lambda I)$ is closed. This follows from the inequality above: let $(A-\lambda I)x_n \to y$ and we want $y = (A-\lambda I)x$ for some x. As $(A-\lambda I)x_n$ converges, $(*)$ implies that $\{x_n\}$ is a Cauchy sequence so it converges to some x. As $A-\lambda I$ is bounded, it follows that $(A-\lambda I)x_n \to (A-\lambda I)x$. Since the limits are unique, $y = (A-\lambda I)x$.

- Third, $\operatorname{image}(A-\lambda I) = H$. Indeed, $\operatorname{image}(A-\lambda I) = \overline{\operatorname{image}(A-\lambda I)} =$

$$= \ker((A-\lambda I)^*)^\perp = \ker(A-\bar{\lambda} I)^\perp = \{0\}^\perp = H$$

this is convex conjugate

by injectivity for all λ with $\operatorname{im}\lambda \neq 0$.

S2 Suppose $A\xi = \lambda\xi$, $A\eta = \mu\eta$ for some $\lambda, \mu \in \mathbb{C}$, $\xi, \eta \in H$. ↑ we know they are real

$$(\xi,\eta) = \frac{1}{\lambda}(\lambda\xi,\eta) = \frac{1}{\lambda}(A\xi,\eta) = \frac{1}{\lambda}(\xi, A\eta) = \frac{1}{\lambda}(\xi,\mu\eta) =$$

$$= \frac{\mu}{\lambda}(\xi,\eta) \Rightarrow \left(1-\frac{\mu}{\lambda}\right)(\xi,\eta) = 0 \Rightarrow (\xi,\eta)=0 \text{ as } \mu \neq \lambda.$$

S3 We have already proved that $(P_M)^* = P_M$. $\Rightarrow$ P_M is self-adjoint.

We study $P_M - \lambda I$.

- Clearly, $\lambda = 0, 1 \in \sigma(P_M)$. Indeed: $\begin{cases} \lambda = 0 & P_M x - 0x = P_M x \text{ has range } M \\ \lambda = 1 & P_M x - x = -P_{M^\perp} x \text{ has range } M^\perp \end{cases}$

as M is closed strict subspace, $0, 1 \in \sigma(P_M)$ (they are in the residual part of the spectrum).

- We claim that $0, 1$ are the only elements of the spectrum. Let $\lambda \neq 0, 1$.

→ $P_M - \lambda I$ is injective. Suppose $P_M x - \lambda x = 0 \Rightarrow P_M x = \lambda x \Rightarrow$

$P_{M^\perp} x = 0$, $P_M x = \lambda P_M x = 0$ as $\lambda \neq 0, 1 \Rightarrow x = 0$.

→ $P_M - \lambda I$ is surjective. Fix $y \in H$. We need to find x s.t. $P_M x - \lambda x = y$

Apply P_M and $P_{M^\perp}$ to this eq. We get $P_M x = \dfrac{P_M y}{1-\lambda}$ and

$P_{M^\perp} x = \dfrac{-P_{M^\perp} y}{\lambda}$. Therefore $x = \dfrac{1}{1-\lambda} P_M y - \dfrac{1}{\lambda} P_{M^\perp} y$ does the job.

S4 $\sigma(M) = [0,1] \subset \mathbb{R}$. □.

$$\langle Mf, g\rangle = \int_0^1 Mf(x)\overline{g(x)}\,dx = \int_0^1 f(x)\,\overline{x\,g(x)}\,dx = \int_0^1 f(x)\,\overline{x\,g(x)}\,dx$$

↑ $x \in \mathbb{R}$

$$= \langle f, Mg\rangle \Rightarrow M = M^*.$$

Similar computation holds true if M is general multiplication operator and multiplier is real-valued.

(C1) Suppose there is a sequence x_n with $\|x_n\| \leqslant 1$ and $\|Tx_n\| \nearrow \infty$.

(we contradict $\exists_C \sup_{\|x\| \leqslant 1} \|Tx\| \leqslant C$). Extract subsequence from $\{Tx_n\}_{n \in \mathbb{N}}$ that converges strongly in norm. Contradiction with $\|Tx_n\| \nearrow \infty$. $\square$.

(C2) (A) $\|x_n\| \leqslant C \ \exists_C \implies \{Tx_n\}$ contains convergent subseq.

(B) $\overline{T(B(0,1))}$ compact

(C) $\overline{T(A)}$ compact for any A bounded.

(C) $\Rightarrow$ (B): obvious.

(B) $\Rightarrow$ (A): let $y_n = \dfrac{x_n}{2C} \in B(0,1)$ so $Ty_n \in T(B(0,1)) \subset \overline{T(B(0,1))}$

Then $\exists_{\{y_{n_k}\} \subset \{y_n\}} \ Ty_{n_k} \longrightarrow z$ for some $z \in X$.

It follows that $Tx_{n_k} = 2C\, Ty_{n_k} \longrightarrow 2C z$.

(A) $\Rightarrow$ (C): Let $\{y_n\} \subset \overline{T(A)}$. For each $n \in \mathbb{N}$, choose $z_n \in T(A)$ so that $\|y_n - z_n\| \leqslant \frac{1}{n}$. As $z_n \in T(A)$ for A bdd, it follows from (A) that z_n contains convergent subsequence; $\{z_{n_k}\}$, i.e. $z_{n_k} \to z$. But $y_{n_k} \to z$ as $\|y_{n_k} - z\| \leqslant \|y_{n_k} - z_{n_k}\| +$

$+ \|z_{n_k} - z\| \leqslant \dfrac{1}{n_k} + \|z_{n_k} - z\| \longrightarrow 0$ as $n_k \to \infty$.

In practical computations, property (A) is the easiest to be verified.

(3) $T: H \to H$ is compact $\Rightarrow$ $\overline{T(B(0,1))}$ compact $\Rightarrow$

as $\forall_{\varepsilon>0}$ we have $\overline{T(B(0,1))} \subseteq \bigcup_{f_\alpha \in \overline{T(B(0,1))}} B(f_\alpha, \varepsilon)$, there is $N(\varepsilon) < \infty$ such

that $\overline{T(B(0,1))} \subseteq \bigcup_{i=1}^{N(\varepsilon)} B(f_i, \varepsilon)$ (finite subcovering).

Let P_ε be projection on the closed linear subspace $\text{span}\{f_1, \dots, f_{N(\varepsilon)}\}$ (closed as it is finite-dimensional). We claim that $T_\varepsilon := P_\varepsilon \circ T$ satisfies $\| P_\varepsilon \circ T - T \| \leq 2\varepsilon$. (or $\| T_\varepsilon - T \| \leq 2\varepsilon$)

Fix $x \in B(0,1)$. There is $i_0 \in \{1, \dots, N_\varepsilon\}$ such that $\| Tx - f_{i_0} \| \leq \varepsilon$.
As $P_\varepsilon f_{i_0} = f_{i_0}$ and $\|P_\varepsilon\| = 1$ (it is projection), we conclude

$$\| T_\varepsilon x - f_{i_0} \| = \| T_\varepsilon x - P_\varepsilon f_{i_0} \| \leq \| Tx - f_{i_0} \| \leq \varepsilon.$$

Therefore, by triangle inequality

$$\| T_\varepsilon x - Tx \| \leq \| T_\varepsilon x - f_{i_0} \| + \| f_{i_0} - Tx \| \leq 2\varepsilon,$$ uniformly in $x \in B(0,1)$.

$\Rightarrow \| T_\varepsilon - T \| \leq 2\varepsilon$.

Remark 1: The opposite direction (if sequence of finite-rank operators $T_i \to T$ in $\mathcal{L}(H,H)$, then T is compact) is in Big Homework 5 (Problem 4).

Remark 2: Using similar arguments, one can check that the same is true for Banach space with Schauder basis. It was a long-standing question (answered in '70s by Per Enflo) whether in every Banach space one can approximate compact operators with finite-rank ones. Per Enflo constructed separable BS without this property; hence also BS which is separable but does not admit Schauder basis.

(C4) $g \in C[0,1]$ fixed

$$T : C[0,1] \to C[0,1] \qquad Tf(x) = \int_0^x f(t) g(t)\, dt$$

Using C2, we start with bdd sequence, $\{f_n\} \subset C[0,1]$, $\|f_n\| \leq C$ and we need to find convergent subsequence in $\{Tf_n\}$ wrt $C[0,1]$.

We use Arzela-Ascoli Theorem:

- equiboundedness: $\|Tf_n\|_\infty \leq 1 \cdot \|f_n\|_\infty \|g\|_\infty \leq C\|g\|_\infty$.
- equicontinuity: $\left(\forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{f_n} \; |f_n(x) - f_n(y)| \leq \varepsilon \text{ if } |x-y| < \delta \right)$

  In particular, if Tf_n are Lipschitz with the same constant (independent of n), this sequence is equicontinuous.

$$|Tf_n(x) - Tf_n(y)| \leq \int_y^x |f_n(t)|\,|g(t)|\, dt \leq |x-y|\, \|f_n\|_\infty \|g\|_\infty \leq$$
$$\leq C \|g\|_\infty |x-y|. \quad \checkmark.$$

So $\{Tf_n\}$ has a subseq. converging in $C[0,1]$.

(C5) $T = Id$ $\quad \overline{T(B(0,1))} = \overline{B(0,1)}$ = compact iff $\dim X < \infty$.

(C6) Hilbert-Schmidt operators $\quad Tf(x) = \int_\Omega K(x,y) f(y)\, dy$

kernel

If K is continuous, it was discussed in the lecture that T is compact.

Now, we assume that $K \in L^2(\Omega \times \Omega)$. Note that this means that for a.e. $x \in \Omega$ $\quad \int_\Omega K^2(x,y)\, dy < \infty$ (i.e. $K(x,\cdot) \in L^2(\Omega)$).

Again, let f_n be bdd in $L^2(\Omega)$. Choose a subsequence of $\{f_n\}$ converging <u>weakly</u> in $L^2(\Omega)$, by Banach-Alaoglu, i.e. $f_{n_k} \rightharpoonup f$.

As $K(x,\cdot) \in L^2(\Omega)$ for a.e. $x \in \Omega$; for a.e. $x \in \Omega$

$$\int_\Omega K(x,y) f_{n_k}(y)\,dy \longrightarrow \int_\Omega K(x,y) f(y)\,dy \quad (\text{a.e. } x \in \Omega).$$

So we obtained $Tf_{n_k} \longrightarrow Tf$ a.e. $x \in \Omega$. We need to upgrade this convergence to $L^2(\Omega)$. Note that

$$|Tf_{n_k}(x)| \leq \int_\Omega |K(x,y)|\,|f_{n_k}(y)|\,dy \overset{\text{Hölder}}{\leq} \|f_{n_k}\|_2 \, \|K(x,\cdot)\|_2 \leq$$

$$\leq C \, \|K(x,\cdot)\|_2 = C\left(\int_\Omega K^2(x,y)\,dy\right)^{1/2}$$

$$\Rightarrow |Tf_{n_k}(x)|^2 \leq \underbrace{C^2 \int_\Omega K^2(x,y)\,dy}$$ $\leftarrow$ integrable majorant

$\Rightarrow$ $Tf_{n_k} \longrightarrow Tf$ in $L^2(\Omega)$ by dominated convergence. $\square$.

(C7) Suppose $0 \notin \sigma(T)$, i.e. T is compact and has bounded inverse.

We know, by Riesz Lemma, that there is a sequence $\{x_k\}$ s.t. $\|x_k\| = 1$ and $\|x_j - x_i\| \geq \frac{1}{2}$ if $i \neq j$. This sequence does not have a convergent subsequence.

T is compact. $\Rightarrow$ Choose convergent subsequence from Tx_k, i.e. $Tx_{n_k} \longrightarrow y$. As T^{-1} is bounded $x_{n_k} = T^{-1}Tx_{n_k} \longrightarrow T^{-1}y$, as $k \to \infty$. Contradiction.

(8) Spectral theory of compact operators

- $0 \in \sigma(T)$
- all other elements of $\sigma(T)$ are eigenvalues
- their multiplicity is finite
- the only possible accumulation point is 0
- $\sigma(T)$ is countable.

This result together with spectral characterization for self-adjoint operators can be used to prove spectral theory for self-adjoint and compact operators.

The proof is performed in few steps.

1) $0 \in \sigma(T)$: we have seen in (7
2) $T-\lambda I$ has closed image
3) To get that all elements of $\sigma(T)$ are eigenvalues, we assume there is $\lambda \in \sigma(T)$ s.t. $T-\lambda I$ is injective but not surjective. Hence, $(T-\lambda I)^n H$ is seq. of decreasing and closed (2) subspaces. Apply Riesz lemma to get contradiction with compactness
4) We check that $\ker(T-\lambda I)^n$ stabilizes for some n and has finite dimension $\Rightarrow$ multiplicity is finite.
5) We let $\varepsilon > 0$ and we prove that there are finitely many eigenvalues λ s.t. $|\lambda| \geq \varepsilon > 0$.
6) Countability of $\sigma(T)$ follows from (5) with $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$.

Nice proof with all details at <u>Wikipedia</u>.

C9 $T: L^2(0,1) \to L^2(0,1)$ $\quad Tf(x) = \int_0^x f(y)\,dy$

T is Hilbert-Schmidt operator with $K(x,y) = \mathbb{1}_{(0,x)}(y) \in L^2((0,1)\times(0,1))$.
Therefore, T is compact. By spectral theory, we know that $0 \in \sigma(T)$ and all other elements are eigenvalues.

Suppose $Tf(x) = \lambda f(x) \Rightarrow \int_0^x f(y)\,dy = \lambda f(x) \Rightarrow f$ is continuous and differentiable. We can write ODE for f:

$$\begin{cases} \lambda f'(x) = f(x) \\ \lambda f(0) = 0 \end{cases} \Rightarrow \begin{cases} \lambda f'(x) = f(x) \\ f(0) = 0 \end{cases} \Rightarrow \begin{cases} f'(x) = \frac{1}{\lambda} f(x) \\ f(0) = 0. \end{cases}$$

This ODE has unique solution (RHS is Lipschitz in f) with $f = 0$.
Therefore, $\sigma(T) = \{0\}$.

$\square$.

C10 We know that $\sigma(G) = \overline{\{g(x) : x \in \mathbb{R}\}}$ where g is bounded and continuous.

If $g = 0$, then $Gf = 0$ and G is compact.
If $g =$ constant, then $Gf =$ constant $\cdot f$ and $0 \notin \sigma(G)$ so G cannot be compact, (constant $\neq 0$).

Finally if g is not constant, there are x, y s.t. $g(x) \neq g(y)$. In particular, by Darboux, g attains all values in $(g(x), g(y))$ and all of them are accumulation points of $\sigma(G)$.
So G cannot be compact.

$\square$

C11

$$\begin{cases} x''(t) = f(t) \\ x(1) = x'(0) = 0 \end{cases} \quad (*) \quad f \in L^2(0,1)$$

Equivalently, integration yields $x'(t) - \overset{0}{\overset{\|}{x'(0)}} = \int_0^t f(s)\,ds \Rightarrow$

$$x(1) - x(u) = \int_u^1 \int_0^t f(s)\,ds\,dt \Rightarrow \boxed{x(u) = -\int_u^1 \int_0^t f(s)\,ds\,dt}$$

(a) Existence: $x(u) = -\int_u^1 \int_0^t f(s)\,ds$ solves the problem.

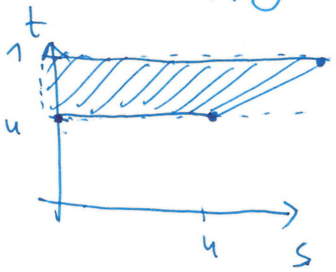
Uniqueness: If x_1, x_2 solve $(*)$, then their difference solve

$$\begin{cases} x''(t) = 0 \\ x(1) = x'(0) = 0 \end{cases} \quad \text{with unique solution } x = 0.$$

Btw, $x \in L^2(0,1)$. $\left| \int_0^1 \int_0^t f(s)\,ds\,dt \right| \underset{\substack{\uparrow \\ \text{Hölder}}}{\le} \left(\int f^2 \right)^{1/2}$ so x is even bounded.

(b) $x := Kf$ or $Kf(u) = -\int_u^1 \int_0^t f(s)\,ds\,dt$

We want to apply Fubini.



$$= -\int_0^1 \int_{\max(s,u)}^1 f(s)\,dt\,ds$$

$$= -\int_0^1 f(s) \underbrace{\left[1 - \max(s,u)\right]}_{-K(u,s)} ds$$

$\Rightarrow Kf(u) = \int_0^1 f(s) K(u,s)\,ds$

$\Rightarrow$ K is integral (or Hilbert-Schmidt operator).

(c) By (b), K is compact.

Spectral theory for compact self-adjoint operators

ST1 $\rightsquigarrow$ Big Homework 5

ST2 Since A is compact and self-adjoint, there is an orthonormal basis of H consisting of eigenvalues of A, $\{\xi_k\}_{k=1}^{\infty}$, $\{\lambda_k\}_{k=1}^{\infty}$ with $\lambda_k \neq 0$ as $\ker A = \{0\}$.

Then, we have $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \quad \forall_{x \in H}$ and this series converges in H (because $S_N := \sum_{i=1}^{N} \langle x, e_i \rangle e_i$, $\|S_N - S_M\|^2 = \sum_{i=N}^{M} \langle x, e_i \rangle^2$ and the latter series converges by Bessel inequality).

We have: $Ax = A \lim_{n \to \infty} \sum_{i=1}^{n} \langle x, e_i \rangle e_i = \lim_{n \to \infty} \sum_{i=1}^{n} \langle x, e_i \rangle A e_i =$

$= \lim_{n \to \infty} \sum_{i=1}^{n} \langle x, e_i \rangle \lambda_i e_i = \sum_{i=1}^{\infty} \langle x, e_i \rangle \lambda_i e_i$ ← this series also converges by the same argument (we use here that $|\lambda_i| \leq \underbrace{\|A\|}_{< \infty}$).

We set $A_n x = \sum_{i=1}^{n} \langle x, e_i \rangle \frac{1}{\lambda_i} e_i$. Then,

$$A_n A x = \sum_{i=1}^{n} \langle Ax, e_i \rangle \frac{1}{\lambda_i} e_i = \sum_{i=1}^{n} \Big\langle \sum_{j=1}^{\infty} \langle x_j, e_j \rangle e_j \lambda_j, e_i \Big\rangle \frac{1}{\lambda_i} e_i$$

$$= \sum_{i=1}^{n} \langle x_i, e_i \rangle e_i \longrightarrow x \quad \text{as } n \to \infty.$$

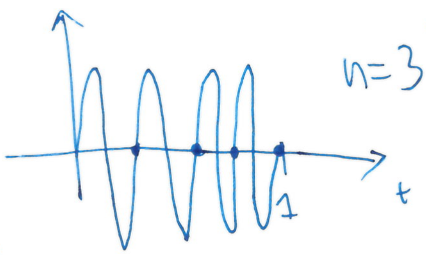
Rademacher system:

$$r_0 = 1, \quad r_m = \mathrm{sgn}\left(\sin\left(2^m \pi t\right)\right) \qquad m \geq 1.$$

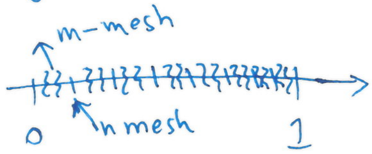
This is orthonormal set:

→ $\int_0^1 1^2 = 1$, $\int_0^1 \mathrm{sgn}^2\left(\sin\left(2^n \pi t\right)\right) = 1$.

→ $\int 1 \cdot \mathrm{sgn}\left(\sin\left(2^n \pi t\right)\right) = 0$ so $\sin(2^n \pi t)$ spends the same time above and below so this integral is 0.



→ $$\int_0^1 \mathrm{sgn}\left(\sin\left(2^n \pi t\right)\right) \mathrm{sgn}\left(\sin\left(2^m \pi t\right)\right) dt \overset{m > m.}{=}$$



$$= \sum_{k=0}^{2^n-1} \int_{k \cdot 2^{-n}}^{(k+1)2^{-n}} \underbrace{\underbrace{\mathrm{sgn}\left(\sin\left(2^n \pi t\right)\right)}_{\text{constant signs}} \underbrace{\mathrm{sgn}\left(\sin\left(2^m \pi t\right)\right)}_{\text{spends the same time above and below}} dt}_{=0}$$

Finally, we need to prove lack of completeness, i.e. $H \neq \overline{\mathrm{span}(r_0, r_1, r_2, \dots)}$.

It is sufficient to check that $\overline{\mathrm{span}(r_0, r_1, r_2, \dots)}$ has nontrivial orthogonal complement, i.e. there is f s.t. $(f, r_i) = 0$ but $f \neq 0$.

We can take $f = r_1 \cdot r_2$. (Justification: ...)