Functional Analysis (WS 19/20)

(midterm review problems and more Hilbert spaces)

Review problems

R1. Consider

$$X = \left\{ f \in C^1(\mathbb{R}) : \int_{-\infty}^{+\infty} |f'(x)|\, dx < \infty \right\}$$

where $C^1(\mathbb{R})$ is the space of continuously differentiable functions on $\mathbb{R}$. Prove that X equipped with a norm

$$\|f\| = |f(0)| + \int_{-\infty}^{+\infty} |f'(x)|\, dx$$

is a normed space. Is it a Banach space?

R2. Let $T : l_1 \to c_0$ be defined with $(Tx)_n = \sum_{k=n}^{\infty} x_k$. Prove that T is a bounded linear operator and compute its norm.

R3. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(0,1)$ such that for all $g \in L^2(0,1)$ we have

$$\lim_{n\to\infty} \int_0^1 f_n(t) g(t)\, dt = 0.$$

Prove that $\sup_n \|f_n\|_2 < \infty$. Is it true that $\lim_{n\to\infty} \|f_n\|_2 = 0$?

R4. Let $y = (y_1, y_2, ...) \in l^1$. Prove that there exists a unique sequence $x = (x_1, x_2, ...) \in l^1$ such that for all $i \in \mathbb{N}$ the following equation is satisfied

$$x_i = \sum_{j=i}^{\infty} \frac{x_j}{2^{j+i}} + y_i.$$

More Hilbert spaces

H1. Prove that if G is a closed subspace of Hilbert space $(H, \langle\cdot,\cdot\rangle)$ then $(G^\perp)^\perp = G$.

H2. Consider subspace G of l^2 consisting of sequences that are nonzero at most on finitely many positions. Compute $G^\perp$ and $(G^\perp)^\perp$. Is G closed in l^2? Recall Schauder basis of l^2.

H3. In this exercise we study space $C[0,1]$ with norm $\|f\| = \left(\int_0^1 |f(t)|^2\, dt\right)^{\frac{1}{2}}$. Is it a Hilbert space with scalar product $\langle f, g\rangle = \int_0^1 f(t)g(t)\, dt$?

H4. In $L^2(0,1)$ consider a subspace V of functions that are constant on $\left[\frac{1}{4}, \frac{3}{4}\right]$. For given $f \in L^2(0,1)$ find explicitly its orthogonal projection on V. Compute subspace $V^\perp$.

H5. Let $(f_n)_{n\in\mathbb{N}}$ be an orthonormal Schauder basis of $L^2(0,1)$. For given $t \in [0,1]$ compute:

$$\sum_{n=1}^{\infty} \left| \int_0^t x^3 f_n(x)\, dx \right|^2.$$

H6. **(midterm May 2016)** In $L^2(-1,1)$ find distance of $f(x) = \frac{1}{x^2+1}$ from the subspace:

$$X = \left\{ f \in L^2(-1,1) : \int_{-1}^1 f(x)\, dx = \int_{-1}^1 x f(x)\, dx = 0 \right\}.$$