Functional Analysis - HW 10

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1 Problem 1

1.1 Description

Let H be a Hilbert space. Below are some simple exercises on orthonormal sets and basis.

(A) Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal set in Hilbert space H. Consider operator $T : H \to c_0$ defined with

$$Tx = \left(\frac{n}{n+1}\langle x, e_n\rangle\right)_{n\in N}.$$

Prove that T is well-defined. Is T a bounded linear operator? If yes, compute its norm.

(B) Let H be an infinite dimensional Hilbert space. Prove that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\|x\| = 1$ and $x_n \rightharpoonup 0$.

(C) Let $y \in l^\infty$, $\{x_n\}_{n\in\mathbb{N}}$ is an orthonormal set in H and $u_n = \frac{1}{n}\sum_{i=1}^{n} e_i y_i$. Prove that $u_n \to 0$ strongly in H.

(D) Let $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ be an orthonormal basis of H. Use density argument to prove that $x_n \rightharpoonup x$ in H if and only if $\langle x_n - x, e_\alpha\rangle \to 0$ for all $\alpha \in \mathcal{A}$ and $\{x_n\}_{n\in\mathbb{N}}$ is bounded in H.

1.2 Solution

(A) We already know that for fixed $x \in H$ we have $\langle x, e_n\rangle \to 0$, so $\frac{n}{n+1}\langle x, e_n\rangle \to 0$ because $|\frac{n}{n+1}| \leq 1$ for all $n \in \mathbb{N}$. That means T is well-defined. We'll prove that T is bounded and its norm is 1. Fix $x \in H$ such that $\|x\| = 1$. By Cauchy-Schwarz inequality and orthonormality of $\{e_n\}_{n\in\mathbb{N}}$ we have

$$\begin{aligned} \|Tx\| &= \sup_{n\in\mathbb{N}} \left| \frac{n}{n+1} \langle x, e_n \rangle \right| \\ &= \sup_{n\in\mathbb{N}} \frac{n}{n+1} |\langle x, e_n \rangle| \\ &\leq \sup_{n\in\mathbb{N}} \frac{n}{n+1} \|x\| \|e_n\| \\ &= \sup_{n\in\mathbb{N}} \frac{n}{n+1} \\ &= 1, \end{aligned}$$

so T is bounded. Now notice that $\|Te_n\| = \frac{n}{n+1}$ and $\|e_n\| = 1$, because $\{e_n\}_{n\in\mathbb{N}}$ is an orthogonal set. With the fact that $\frac{n}{n+1} \to 1$ as $n \to \infty$, we see that the norm of T is indeed equal to 1.

(B) Let $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ be an orthonormal basis of H. As H is infinite dimensional, the basis has infinitely many elements, so we can choose a countable subset $\{e_n\}_{n\in\mathbb{N}}$. We know that $\langle x, e_n \rangle \to 0$ for every $x \in H$, but we also know that every functional in H^* is a scalar product with some $x \in H$, so $e_n \rightharpoonup 0$.

(C) Let $M = \sup_{n\in\mathbb{N}} |y_n|$. Obviously $M < \infty$, as $y \in l^\infty$. Using orthonormality of $\{x_n\}_{n\in\mathbb{N}}$ and linearity of scalar product we estimate

$$\begin{aligned} \|u_n\|^2 &= \langle u_n, u_n \rangle \\ &= \left\langle \frac{1}{n} \sum_{i=1}^{n} e_i y_i, \frac{1}{n} \sum_{i=1}^{n} e_i y_i \right\rangle \\ &= \frac{1}{n^2} \sum_{i=1}^{n} y_i^2 \\ &\leq \frac{M^2}{n}. \end{aligned}$$

Taking $n \to \infty$ we obtain $\|u_n\| \to 0$, so $u_n \to 0$ strongly in H.

(D) ($\Longrightarrow$) Let $x_n \rightharpoonup x$. Weakly converging sequences are bounded, so $(x_n)_{n\in\mathbb{N}}$ is bounded. We know that every functional in H^* is a scalar product with some $x \in H$, so we have $\langle x_n, v \rangle \to 0$ for every $x \in H$. Hence $\langle x_n, e_n \rangle \to 0$ and we are done.

($\Longleftarrow$) We'll again use the fact that every functional in H^* is a scalar product with some $x \in H$. We'll start with proving that $\langle x_n - x, v\rangle \to 0$ holds for all v in $\operatorname{span}(\{x_\alpha\}_{\alpha\in\mathcal{A}})$. Let $v = \sum_{\alpha\in\mathcal{A}} c_\alpha e_\alpha$, then

$$\langle x_n - x, v\rangle = \left\langle x_n - x, \sum_{\alpha\in\mathcal{A}} c_\alpha e_\alpha \right\rangle = \sum_{\alpha\in\mathcal{A}} c_\alpha \langle x_n - x, e_\alpha\rangle ,$$

where we used the linearity of scalar product. But as $\langle x_n - x, e_\alpha\rangle \to 0$ and the linear combinations in $\operatorname{span}(\{x_\alpha\}_{\alpha\in\mathcal{A}})$ are finite,

$$\sum_{\alpha\in\mathcal{A}} c_\alpha \langle x_n - x, e_\alpha\rangle \to 0$$

as $n \to \infty$, so $\langle x_n - x, v\rangle \to 0$. Now let $y \in H$. We know that $H = \overline{\operatorname{span}(\{x_\alpha\}_{\alpha\in\mathcal{A}})}$, so there exists sequence $(y_n)_{n\in\mathbb{N}} \in \operatorname{span}(\{x_\alpha\}_{\alpha\in\mathcal{A}})$ such that $y_n \to y$ strongly in H. Now we can estimate

$$\begin{aligned}
|\langle x_n - x, y\rangle| &= |\langle x_n - x, y\rangle - \langle x_n - x, y_k\rangle + \langle x_n - x, y_k\rangle| \\
&\le |\langle x_n - x, y\rangle - \langle x_n - x, y_k\rangle| + |\langle x_n - x, y_k\rangle| \\
&= |\langle x_n - x, y - y_k\rangle| + |\langle x_n - x, y_k\rangle| \\
&\le \|x_n - x\| \|y - y_k\| + |\langle x_n - x, y_k\rangle| \\
&\le (\|x_n\| + \|x\|) \|y - y_k\| + |\langle x_n - x, y_k\rangle| .
\end{aligned}$$

As $(x_n)_{n\in\mathbb{N}}$ is bounded, there exist $M \in \mathbb{R}$ such that $\|x_n\| + \|x\| \le M$ for all $n \in \mathbb{N}$. Hence

$$|\langle x_n - x, y\rangle| \le M \|y - y_k\| + |\langle x_n - x, y_k\rangle| = M \|y - y_k\| ,$$

and by taking $n \to \infty$ we get

$$|\langle x_n - x, y\rangle| \le M \|y - y_k\| .$$

Now it suffices to take $k \to \infty$ to get $|\langle x_n - x, y\rangle| = 0$, which ends the proof.

2 Problem 2

2.1 Description

Let μ be a Gaussian measure, i.e. a measure with density $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Let $H = L^2(\mathbb{R}, \mu)$.

(A) Let $X = \operatorname{span}(1, x, x^2)$. Prove that X is a linear subspace of H.

(B) Recall Gram-Schmidt algorithm. Use it to find an orthonormal basis of X.

(C) Compute the distance of $f(x) = |x|$ from X.

2.2 Solution

Let us denote $g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and let N be a random variable with distribution $\mathcal{N}(0,1)$. We will use the fact that $\mathbb{E}X^1 = \mathbb{E}X^3 = 0$, $\mathbb{E}X^2 = 1$, $\mathbb{E}X^4 = 3$, $\mathbb{E}|X| = \sqrt{2/\pi}$ and $\mathbb{E}|X|^3 = 2\sqrt{2/\pi}$.

(A) Obviously $\mathrm{span}(1, x, x^2)$ contains 0 and is closed under addition and multiplication by scalars. Therefore, we only need to show $X \subset H$. Let us take any $v \in X$. We can write $v(x) = a + bx + cx^2$ for some $a, b, c \in \mathbb{R}$. We have to show $\int_{\mathbb{R}} v(x)^2 g(x) dx < \infty$, but $v(x)^2 = a_0 + \ldots + a_4 x^4$ for some $a_0, \ldots, a_4 \in \mathbb{R}$, so $\int_{\mathbb{R}} v(x)^2 g(x) dx = a_0 + \ldots + a_4 \mathbb{E}N^4 < \infty$, as all the moments of N are finite.

(B) We will find orthogonal basis (e_1, e_2, e_3). First, we have:

$$\|1\|_H^2 = \langle 1, 1 \rangle = \int_{\mathbb{R}} g(x) dx = 1,$$

so we can denote $e_1 = 1$. Second, we can compute

$$\langle 1, x \rangle = \int_{\mathbb{R}} x g(x) dx = \mathbb{E}N = 0,$$

$$\|x\|_H^2 = \int_{\mathbb{R}} x^2 g(x) dx = \mathbb{E}N^2 = 1,$$

so we can take $e_2 = x$. That was quite easy so far, but things get a little bit more complicated when it comes to e_3. We can compute:

$$\langle 1, x^2 \rangle = \int_{\mathbb{R}} x^2 g(x) dx = \mathbb{E}N^2 = 1$$

and

$$\langle x, x^2 \rangle = \int_{\mathbb{R}} x^3 g(x) dx = \mathbb{E}N^3 = 0,$$

so according to Gram-Schmidt algorithm we should take $e_3 = (x^2 - 1)/\left\|x^2 - 1\right\|_H$. Clearly:

$$\left\|x^2 - 1\right\|_H^2 = \int_{\mathbb{R}} (x^2 - 1)^2 g(x) dx = \int_{\mathbb{R}} (x^4 - 2x^2 + 1) g(x) dx =$$
$$= \mathbb{E}N^4 - 2\mathbb{E}N^2 + 1 = 3 - 2 + 1 = 2,$$

so we take $e_3 = \frac{1}{\sqrt{2}}(x^2 - 1)$.

(C) As we have an orthonormal basis of X it is easy to find the distance from f to X. Indeed, we can express f as a sum of its projections on e_1, e_2, e_3 and the part which is orthogonal to X. To compute the norm of the orthogonal part we can use the Pythagorean theorem. Clearly, this norm is the distance we need as X is convex and closed. To sum up:

$$\mathrm{dist}(f, X)^2 = \|f\|_H^2 - \langle f, e_1\rangle^2 - \langle f, e_2\rangle^2 - \langle f, e_3\rangle^2.$$

Now step by step:

$$\|f\|_H^2 = \int_{\mathbb{R}} |x|^2 g(x)dx = \int_{\mathbb{R}} x^2 g(x)dx = \mathbb{E}N^2 = 1,$$

$$\langle f, 1\rangle = \int_{\mathbb{R}} |x| g(x)dx = \mathbb{E}|N| = \sqrt{\frac{2}{\pi}},$$

$$\langle f, x\rangle = \int_{\mathbb{R}} |x| x g(x)dx = 0,$$

(because $|x|xg(x)$ is odd function and its absolute value is bounded by $x^2 g(x)$)

$$\langle f, x^2\rangle = \int_{\mathbb{R}} |x| x^2 g(x)dx = \int_{\mathbb{R}} |x|^3 g(x)dx = \mathbb{E}|N|^3 = 2\sqrt{\frac{2}{\pi}},$$

and finally:

$$\langle |x|, \frac{1}{\sqrt{2}}(x^2 - 1)\rangle = \frac{1}{\sqrt{2}}(\langle |x|, x^2\rangle - \langle |x|, 1\rangle) = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{\pi}} = \frac{1}{\sqrt{\pi}}.$$

Now we can write:

$$\mathrm{dist}(f, X) = \sqrt{1 - \frac{2}{\pi} - \frac{1}{\pi}} = \sqrt{1 - \frac{3}{\pi}} \approx 0.21$$

and our job is finished.