AF-2-Homework

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1. For cases (a) and (b) we have one and the same counterexample to show that these are not normed spaces, which also means they are not Banach spaces. We take $f(x) = 1$, which is clearly not equal to 0, but if we calculate the respective norms, they turn out to be 0 which is a direct contradiction with the definition of a norm. Now we are going to look at case (c). Here we also want to show that this is not a normed space and we are going to achieve that by calculating the norm of the function $f(x) = \sqrt{x}$. Firstly we need to show that $f \in C^{\frac{1}{2}}[0,1]$. This will be the case, because for $1 \geq x > y \geq 0$ we have

$$\frac{|\sqrt{x} - \sqrt{y}|}{\sqrt{|x-y|}} = \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x-y}} = \frac{x-y}{\sqrt{x-y}(\sqrt{x} + \sqrt{y})} = \frac{\sqrt{x-y}}{\sqrt{x} + \sqrt{y}} \leq 1$$

Last inequality follows from the fact that $\sqrt{x-y} \leq \sqrt{x}$ and $\sqrt{x} + \sqrt{y} \geq \sqrt{x}$. Thus we have proven that f has to have a finite $|\cdot|_{\frac{1}{2}}$ norm, which makes it a member of $C^{\frac{1}{2}}[0,1]$, but $|f|_{LIP} = \infty$, which implies that this is not a normed space. This counterexample also works for the space given in (e), since the "norm" of $\sqrt{x}$ in that space will also be infinite, because of the $|f|_{LIP}$ term. Moving on to (d), we will show that this space is in fact a Banach space. Firstly we check the usual properties of a norm:

- $||f||_\infty + |f|_{\frac{1}{2}} = 0 \iff f = 0$ follows from the fact that $||\cdot||_\infty$ is a norm.
- $||\lambda f||_\infty + |\lambda f|_{\frac{1}{2}} = |\lambda|(||f||_\infty + |f|_{\frac{1}{2}})$ follows from the fact that $||\cdot||_\infty$ and $|\cdot|$ are norms
- It suffices to check that $|\cdot|_{\frac{1}{2}}$ holds the triangle inequality. We check it directly using the triangle inequality for $|\cdot|$:

$$\frac{|f(x) + g(x) - f(y) - g(y)|}{\sqrt{|x-y|}} \leq \frac{|f(x) - f(y)|}{\sqrt{|x-y|}} + \frac{|g(x) - g(y)|}{\sqrt{|x-y|}}$$

Now, if we take the supremum on both sides the inequality still holds and the right hand side becomes the sum of supremums, because terms summed are ≥ 0, so we get the triangle inequality for our function $|\cdot|_{\frac{1}{2}}$.

Thus our space $(C^{\frac{1}{2}}[0,1], ||\cdot||_\infty + |\cdot|_{\frac{1}{2}})$ is a normed space. Now to show this is a Banach space we, as usual, take a Cauchy sequence $(f_n) \subset C^{\frac{1}{2}}[0,1]$. From the definition of a Cauchy sequence we can quickly deduce that for all $\varepsilon > 0$ we have

- $\exists N_\varepsilon \; \forall n, k > N_\varepsilon \; ||f_n - f_k||_\infty < \varepsilon$
- $\exists M_\varepsilon \; \forall m, l > M_\varepsilon \; |f_m - f_l|_{\frac{1}{2}} < \varepsilon$

The fact that (f_n) is a Cauchy sequence in $C^0[0,1]$ gives us a candidate for the limit in our space $(C^{\frac{1}{2}}[0,1], ||\cdot||_\infty + |\cdot|_{\frac{1}{2}})$ and it will be its limit in $(C^0[0,1], ||\cdot||_\infty)$, we will call it f. We have to show that:

(a) $f \in C^{\frac{1}{2}}[0,1]$

(b) $||f_n - f||_\infty + |f_n - f|_{\frac{1}{2}} \to 0$, here obviously it suffices to say that $|f_n - f|_{\frac{1}{2}} \to 0$, since we know that $||f_n - f||_\infty \to 0$ by construction.

To show (a) we to are going to look at a expression:

$$|f_n(x) - f_n(y)| \leq |f|_{\frac{1}{2}} \sqrt{|x-y|} \tag{1}$$

This follows from the definition of $|\cdot|_{\frac{1}{2}}$ and the fact that for all n, $f_n \in C^{\frac{1}{2}}[0,1]$ makes the $|f_n|_{\frac{1}{2}}$ finite for all n. We can look at a sequence $(|f_n|_{\frac{1}{2}})$ as a sequence in $\mathbb{R}$, where it will also be Cauchy. That means there exists a limit of this sequence in $(\mathbb{R}, |\cdot|)$, we are going to call it a. Now if we let $n \to \infty$ in (1) we will see that

$$|f(x) - f(y)| \leq a\sqrt{|x-y|} \iff \frac{|f(x) - f(y)|}{\sqrt{x-y}} \leq a \iff |f|_{\frac{1}{2}} \leq a < \infty$$

Thus $f \in C^{\frac{1}{2}}[0,1]$. We will now prove the last part - that $|f - f_n|_{\frac{1}{2}} \to 0$:

$$|f - f_n|_{\frac{1}{2}} = \sup_{x \neq y} \frac{|f(x) - f_n(x) - (f(y) - f_n(y))|}{\sqrt{|x-y|}} =$$
$$= \sup_{x \neq y} \liminf_{k \to \infty} \frac{|f_k(x) - f_n(x) - (f_k(y) - f_n(y))|}{\sqrt{|x-y|}} \leq$$
$$\leq \liminf_{k \to \infty} \sup_{x \neq y} \frac{|f_k(x) - f_n(x) - (f_k(y) - f_n(y))|}{\sqrt{|x-y|}} = \liminf_{k \to \infty} |f_n - f_k|_{\frac{1}{2}}$$

If we let $n \to \infty$, then RHS$\to 0$ from the fact that (f_n) is a Cauchy sequence in our space. Thus we have proven that $|f - f_n| \to 0$ completing the proof that $(C^{\frac{1}{2}}[0,1], ||\cdot||_\infty + |\cdot|_{\frac{1}{2}})$ is a Banach space.

2. We will start off by proving that $(l^p, ||\cdot||_p)$ is a Banach space using the a theorem, which was proven during lectures - that for any $(X, \mathcal{F}, \mu)$, with μ being σ - finite, the space $L^p(X, \mathcal{F}, \mu)$ is a Banach space for all $p \in [1, \infty]$. If we set $X := \mathbb{N}$, $\mathcal{F} := \mathcal{P}(\mathbb{N})$ and $\mu := \sum_{n=1}^{\infty} \delta_n$, where δ_n is a Dirac measure ($\delta_n(A) = 1$ if $n \in A$ and 0 otherwise). μ is a σ - finite, because we can take $K_n = \{n\}$, then $X = \bigcup_{n=1}^{\infty}$, with $\mu(K_n) = 1 < \infty$, which means that μ is in fact σ - finite. Therefore $L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ is a Banach measure. Is this space the same as $(l^p, ||\cdot||_p)$? To see that, let $f \in L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, that means that $f : \mathbb{N} \to \mathbb{R}$ and we know that the value of $\int_{\mathbb{N}} |f|^p d\mu = \sum_{n=1}^{\infty} |f(n)|$ is finite, therefore $L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) = (l^p, ||\cdot||_p)$, since every functions from $\mathbb{N}$ to $\mathbb{R}$ can be assigned exactly one sequence and vice versa. If $p = \infty$, then it is clear that $||f||_{L^\infty} = ||f(n)||_\infty$, because in sets of measure 0 with respect to μ are empty sets. Therefore

$$||f||_{L^\infty} = ess \sup |f| = \sup |f| = ||f(n)||_\infty$$

Thus $L^\infty(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) = (l^\infty, ||\cdot||_\infty)$. Moving on to (b), to prove that $\varphi \in (l^1)^*$ we need to check that φ satisfies two conditions:

1. φ is linear
2. φ is bounded

Ad 1.
Let $\alpha, \beta \in \mathbb{R}$ and $u, v \in l^1$, we have

$$\varphi(\alpha u + \beta v) = \sum_{n=1}^{\infty} \frac{1}{2^n}(\alpha u_n + \beta v_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} \alpha u_n + \sum_{n=1}^{\infty} \frac{1}{2^n} \beta v_n = \alpha\varphi(u) + \beta\varphi(v)$$

Here we are using the fact that since $u, v \in l^1$ we know that the series $\sum u_n$ and $\sum v_n$ are absolutely convergent and that makes the series of $\varphi(u)$ absolutely convergent for all $u \in l^1$. That means φ is well defined (since rearranging terms will not change the value of $\varphi(u)$) and also that the equations above hold. Ad 2.
This quickly follows from the fact that for all n we have

$$\frac{1}{2^n}|v_n| \le |v_n|$$

Which implies that

$$|\varphi(v)| = \left|\sum_{n=1}^{\infty} \frac{1}{2^n} v_n\right| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} |v_n| \leq ||v||_1$$

Therefore φ is bounded. Thus we have shown that $\varphi \in (l^1)^*$. Lastly we will calculate the operator norm $||\varphi||$:

$$||\varphi|| = \sup_{||u||_1 = 1} |\varphi(u)|$$

We can notice that for any u such that $||u||_1 = 1$:

$$|\varphi(u)| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} |u_n| = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} |u_n| \leq \frac{1}{2} \sum_{n=1}^{\infty} |u_n| = \frac{1}{2}$$

Which also makes the supremum bounded by $\frac{1}{2}$. Now if we set $u = (1, 0, \dots)$, we have $\varphi(u) = \frac{1}{2}$, which means that our upper bound is attainable, this automatically makes it the supremum of $|\varphi|$, since it has to be the least upper bound of $|\varphi|$. Therefore $||\varphi|| = \frac{1}{2}$