Functional Analysis - HW 4

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1 Problem 1

1.1 Description

Let $(X, \|\cdot\|_X)$ be a Banach space. Consider a linear operator $T : X \to X^*$ such that for all $x \in X$:

$$(Tx)(x) \geq 0.$$

Prove that T is a bounded linear operator, i.e. $T \in \mathcal{L}(X, X^*)$.

1.2 Solution

We will prove that the graph of T is closed, which implies its boundedness (by the closed graph theorem). Let us take any $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $Tx_n \to E \in X^*$. We need to prove $Tx = E$. Let us take any $z \in X$ and denote $C_z := (Tz)(z) \geq 0$. Let us also fix a constant $\varepsilon > 0$. We have:

$$(T(x - x_n + \varepsilon z))(x - x_n + \varepsilon z) \geq 0,$$

which is equivalent to:

$$(Tx)(x-x_n)-(Tx_n)(x-x_n)+\varepsilon T(z)(x-x_n)+\varepsilon(T(x-x_n))(z)+\varepsilon^2(Tz)(z) \geq 0.$$

The equation above is justified as both T and Tx for any $x \in X$ are linear. We can further transform this inequality to:

$$(Tx)(x - x_n) - (Tx_n - E)(x - x_n) - E(x - x_n)+$$

$$+\varepsilon T(z)(x - x_n) + \varepsilon(T(x - x_n))(z) + \varepsilon^2 C_z \geq 0.$$

Now, knowing that $Tx_n \to E$ and $x_n \to x$, we can take $n \to \infty$ and obtain

$$\varepsilon(T(x)(z) - E(z)) + \varepsilon^2 C_z \geq 0 \iff T(x)(z) - E(z) + \varepsilon C_z \geq 0.$$

After taking $\varepsilon \to 0$ we get

$$T(x)(z) - E(z) \geq 0.$$

But this inequality holds for every $z \in X$, so we also have

$$0 \leq T(x)(-z) - E(-z) = -T(x)(z) + E(z) \iff T(x)(z) - E(z) \leq 0,$$

thus $T(x)(z) = E(z)$ for every $z \in X$ and we are done.

2 Problem 2

2.1 Description

We write c_0 for the space of sequences $x = (x_1, x_2, \dots)$ such that $\lim_{n\to\infty} x_n = 0$ (i. e. sequences converging to 0). Space c_0 is equipped with the usual supremum norm $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.

- Prove that $c_0 \subset l^\infty$.
- Prove that $(c_0, \|\cdot\|_\infty)$ is a Banach space.
- Let $z = (z_1, z_2, \dots)$ be a sequence of real numbers such that whenever $y = (y_1, y_2, \dots) \in c_0$, we have that $\sum_{n\geq 1} z_n y_n$ is convergent in $\mathbb{R}$. Prove that $\sum_{n\geq 1} |z_n|$ is convergent.

  Hint: for $y \in c_0$, consider $\varphi_k \in (c_0)^*$ defined with $\varphi_k(y) = \sum_{n=1}^{k} z_n y_n$.

2.2 Solution

First of all, every convergent sequence is bounded, so $c_0 \subset l^\infty$. We know that $(l^\infty, \|\cdot\|_\infty)$ is a Banach space, so to prove that $(c_0, \|\cdot\|_\infty)$ is a Banach space we only need to prove that c_0 is closed. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence of sequences from c_0 such that $\lim_{n\to\infty} x_n = x \in l^\infty$. We will denote $x_n = (x_{n1}, x_{n2}, \dots)$ and $x = (x_1, x_2, \dots)$. Suppose that $x \notin c_0$, then there exists $\varepsilon > 0$ such that for some index sequence $(i_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} i_n = \infty$ we have $|x_{i_n}| > \varepsilon$ for all $n \in \mathbb{N}$. Now fix $\delta < \varepsilon$. There exists n_δ such that $\|x - x_{n_\delta}\|_\infty < \delta$. For every $k \in \mathbb{N}$ we can estimate

$$|x_{n_\delta i_k}| \geq |x_{i_k}| - |x_{i_k} - x_{n_\delta i_k}| \geq \varepsilon - \delta,$$

which contradicts the fact that $x_{n_\delta} \in c_0$. Hence, $x \in c_0$, so c_0 is a closed subset of l^∞, so c_0 is a Banach space.

We'll proceed to the proof of the final statement. As written in the hint, we consider $\varphi_k \in (c_0)^*$ defined with $\varphi_k(y) = \sum_{n=1}^{k} z_n y_n$. For $y \in c_0$ we know that $\sum_{n\geq 1} z_n y_n$ is convergent in $\mathbb{R}$, so the sequence of partial sums is bounded. Therefore, for all $y \in c_0$ we have $\sup_{k\in\mathbb{N}} |\varphi_k(y)| < \infty$. Using Banach-Steinhaus theorem we obtain $\sup_{k\in\mathbb{N}} \|\varphi_k\| < \infty$, which means that for every $k \in \mathbb{N}$ and $y \in c_0$ such that $\|y\|_\infty = 1$ we have $|\phi_k(y)| < C$ for some constant $C > 0$. By taking $y_k = (\mathrm{sgn}(z_1), \mathrm{sgn}(z_2), \dots, \mathrm{sgn}(z_k), 0, 0, \dots)$ we get

$$C > |\phi_k(y_k)| = |\sum_{n=1}^{k} z_n y_n| = \sum_{n=1}^{k} |z_n|$$

for every $k \in \mathbb{N}$, so $\sum_{n\geq 1} |z_n|$ is convergent.