# Functional Analysis, PS5

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**(A1)** Scalar product on $H$:

(1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(2) linear in its first argument

(3) $\langle x, x \rangle > 0 \quad \forall_{x \neq 0}$, $\langle x, x \rangle = 0 \quad x = 0$

$L^2(0,1)$ over $\mathbb{R}$ $\quad \langle f, g \rangle = \int_0^1 f(x) g(x) \, dx$

$L^2(0,1)$ over $\mathbb{C}$ $\quad \langle f, g \rangle = \int_0^1 f(x) \overline{g(x)} \, dx$

$\ell^2$ over $\mathbb{R}$ $\quad \langle x, y \rangle = \sum x_i y_i \quad (x_i, y_i \in \mathbb{R})$

$\ell^2$ over $\mathbb{C}$ $\quad \langle x, y \rangle = \sum x_i \overline{y_i} \quad (x_i, y_i \in \mathbb{C})$.

**(A2)** $\langle f, g \rangle = \int_0^1 f(x) g(x) \, dx$ is a scalar product on $C[0,1]$ but $(C[0,1], \langle \cdot, \cdot \rangle)$ is not a Hilbert space because $(C[0,1], \|\cdot\|_2)$ is not a Banach space.

**A3**

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle$$
$$= \overline{\langle x+y, x\rangle} + \overline{\langle x+y, y\rangle} =$$
$$= \overline{\langle x, x\rangle} + \overline{\langle y, x\rangle} + \overline{\langle x, y\rangle} + \overline{\langle y, y\rangle}$$
$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \overline{\langle y, y\rangle}$$
$$= \|x\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle + \|y\|^2.$$

$\square$.

**A4**

$$\langle f, g\rangle = \left\langle f(t)e^{t/2}, g(t)e^{t/2} \right\rangle$$

**A5**

$$\forall_{x,y\in H} \quad 2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2 \iff$$
$$\frac{1}{2}\left(\|x\|^2 + \|y\|^2\right) = \left\|\frac{x+y}{2}\right\|^2 + \left\|\frac{x-y}{2}\right\|^2$$

Let $x, y$ be s.t. $\|x\| = \|y\| = 1$, $\|x-y\| \geq \varepsilon$. Then,

$$\left\|\frac{x+y}{2}\right\|^2 \cancel{\leq} \frac{1}{2}\left(\|x\|^2 + \|y\|^2\right) - \left\|\frac{x-y}{2}\right\|^2 \cancel{\leq}$$
$$\cancel{\leq} 1 - \left\|\frac{x-y}{2}\right\|^2 \leq 1 - \frac{\varepsilon^2}{4} = (1-\delta)^2$$

$\Rightarrow$ we choose $\delta = 1 - \left(1 - \frac{\varepsilon^2}{4}\right)^{1/2}$.

Mention here Milman-Pettis Theorem: if $E$ unif. convex $\Rightarrow E^{**} = E$ (geometry $\Rightarrow$ analysis)

# PROJECTIONS, ORTH. COMPLEMENTS, ETC...

**B1** $K \subset H$ (just a subset).

$K^{\perp}$ is a linear subspace : if $u, \tilde{u} \in K^{\perp}$, $\alpha u + \beta \tilde{u} \in K^{\perp}$

as $\langle \alpha u + \beta \tilde{u}, v \rangle = 0 = \alpha \langle u, v \rangle + \beta \langle \tilde{u}, v \rangle = 0$.

$K^{\perp}$ is closed : If $\{v_n\} \subset K^{\perp}$, $v_n \to v$ we have

$$0 = \langle v_n, x \rangle \quad \forall_{x \in H} \Rightarrow \langle v, x \rangle = 0 \Rightarrow v \in K^{\perp}.$$

**B3** $X = \{ f \in L^2(-1,1) : f(x) = f(-x) \}$

(a) $X$ is closed : $\{f_n\} \subset X$, $f_n \to f$ in $L^2(0,1)$

$$\left( \int_{-1}^{1} |f(x) - f(-x)|^2 \right)^{1/2} \leq \left( \int_{-1}^{1} |f_n(x) - f_n(-x)|^2 \right)^{1/2}$$

$$+ \left( \int_{-1}^{1} |f(x) - f_n(x)|^2 dx \right)^{1/2} + \left( \int_{-1}^{1} |f(-x) - f_n(-x)|^2 dx \right)^{1/2}$$

$\to 0$ as $n \to \infty$ $\Rightarrow$ $f(x) = f(-x)$ a.e. $x \in (-1,1)$.

(b) $X^{\perp} = \{ f \in L^2(-1,1) : \langle f, g \rangle = 0 \ \forall_{g \in X} \}$.

$$0 = \langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx =$$

$$= \int_{-1}^{0} f(x) g(x)\, dx + \int_{0}^{1} f(x) g(x)\, dx =$$

$$= \int_0^1 f(-x) g(-x)\, dx + \int_0^1 f(x) g(x)\, dx =$$

$$= \int_0^1 [f(-x) + f(x)]\, g(x)\, dx$$

So we have characterization:

$$f \in X^{\perp} \iff \int_0^1 [f(-x) + f(x)]\, g(x)\, dx = 0 \ \forall_{g \in X}.$$

We claim $X^{\perp} = \{ f \in L^2(-1,1) : f(-x) + f(x) = 0 \}$.

$\subseteq$: $f \in X^{\perp}$. i.e. $\int_0^1 (f(-x) + f(x))\, g(x) = 0 \ \forall_{g \in X}$

Take $g(x) = f(x) + f(-x) \in X$. $\Rightarrow \int_0^1 (f(-x) + f(x))^2 = 0$

$\Rightarrow f(-x) + f(x) = 0$ p.w.

$\supseteq$: If $f(x)+f(-x)=0$ the condition is trivially satisfied.

(c) $P_X f,\ P_{X^\perp} f = ?$

$$f = P_X f + P_{X^\perp} f$$

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\in X} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\in X^\perp}.$$

By uniqueness $P_X f = \frac{f(x)+f(-x)}{2}$, $P_{X^\perp} f = \frac{f(x)-f(-x)}{2}$.

B4 Find polynomial $w(t)$, $\deg(w)\leq 2$ s.t.

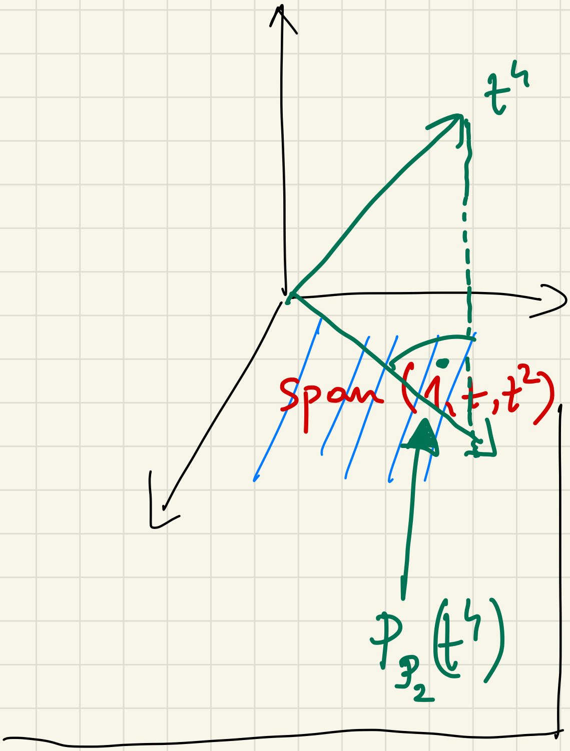
$\int |w(t)-t^4|^2\,dt$ is the smallest.

$$\int_0^1 |w(t)-t^4|^2\,dt = \|w(t)-t^4\|_2^2$$

Let $\mathcal{P}_2$ = space of polynomials with degree $\leq 2$.

We need to find

$$\inf_{w\in\mathcal{P}_2} \|w(t)-t^4\|_2^2 \rightsquigarrow$$ this is attained by $w(t) = P_{\mathcal{P}_2} t^4$.




Let $g(t) = P_{\mathcal{P}_2}(t^4)$.

$$\begin{cases} t^4 - g(t) \perp 1 \\ t^4 - g(t) \perp t \\ t^4 - g(t) \perp t^2 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \int_0^1 (t^4 - g(t)) \cdot 1 = 0 \\ \int_0^1 (t^4 - g(t)) \cdot t = 0 \\ \int_0^1 (t^4 - g(t))\, t^2 = 0 \end{cases}$$

$$g(t) = a + bt + ct^2$$

$$\Rightarrow \begin{cases} \frac{1}{5} - a - \frac{b}{2} - \frac{c}{3} = 0 \\ \frac{1}{5} - \frac{a}{2} - \frac{b}{3} - \frac{c}{4} = 0 \\ \frac{1}{5} - \frac{a}{3} - \frac{b}{4} - \frac{c}{5} = 0 \end{cases} \Rightarrow (a,b,c) = \ldots$$

The result is $g(t) = a + bt + ct^2$

**Alternative way:** Compute first orthogonal basis of $\mathcal{P}_2$.

B6

$$E = \left\{ f \in L^2(0,1) : \int_0^1 f(t) = 0, \int_0^1 f(t)\, t = 0 \right\}.$$

$$= \left\{ f \in L^2(0,1) : \langle f, 1 \rangle = \langle f, t \rangle = 0 \right\}.$$

$$\Rightarrow E = \left( \mathrm{span}(1,t) \right)^{\perp}.$$

We want to compute $\inf_{f \in E} \int_0^1 |t^3 - f(t)|^2 dt =$

$$= \int_0^1 |t^3 - P_E(t^3)|^2 dt$$

It's not so easy to compute $P_E(t^3)$ directly but we can compute $P_K(t^3)$, $K = \mathrm{span}(1,t)$. Then,

$$P_E(t^3) = t^3 - P_K(t^3).$$

$P_K(t^3) = ?$ We introduce $g(t) = P_K(t^3) = a + bt$.

$$\begin{cases} t^3 - g(t) \perp 1 \\ t^3 - g(t) \perp t \end{cases} \Rightarrow \begin{cases} \int_0^1 t^3 - a - bt = 0 \\ \int_0^1 (t^3 - a - bt)t = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{1}{4} - a - \frac{b}{2} = 0 \\ \frac{1}{5} - \frac{a}{2} - \frac{b}{3} = 0 \quad | \cdot 2 \end{cases} \Rightarrow \begin{cases} \frac{1}{4} - a - \frac{b}{2} = 0 \\ \frac{2}{5} - a - \frac{2}{3}b = 0 \end{cases}$$

$$\begin{cases} \frac{1}{4} - a - \frac{b}{2} = 0 \\ \frac{2}{5} - a - \frac{2}{3}b = 0 \end{cases} \Rightarrow \frac{1}{4} - \frac{2}{5} + \frac{2}{3}b - \frac{b}{2} = 0 \Rightarrow$$

$$\Rightarrow \frac{5}{20} - \frac{8}{20} + \frac{4}{6}b - \frac{3}{6}b = 0 \Rightarrow$$

$$\Rightarrow -\frac{3}{20} + \frac{1}{6}b \Rightarrow b = \frac{18}{20} = \frac{9}{10}.$$

$$a = \frac{1}{4} - \frac{b}{2} = \frac{1}{4} - \frac{9}{20} = \frac{5}{20} - \frac{9}{20} = -\frac{4}{20} = -\frac{1}{5} \Rightarrow a = -\frac{1}{5}$$

$$\Rightarrow g(t) = P_K(t^3) = -\frac{1}{5} + \frac{9}{10}t.$$

$$\Rightarrow P_E(t^3) = P_{K^\perp}(t^3) = t^3 - P_K(t) = t^3 - g(t).$$

$$\int_0^1 |t^3 - P_E(t^3)|^2 dt = \int_0^1 |g(t)|^2 dt =$$

$$= \int_0^1 \left[\frac{1}{25} + \frac{81}{100}t^2 - \frac{2 \cdot 9}{50}t\right] dt =$$

$$= \frac{1}{25} + \frac{\cancel{81}^{27}}{100} \cdot \frac{1}{\cancel{3}} - \frac{18}{50}\,\frac{1}{2} = \frac{4 + 27 - 18}{100} = \frac{13}{100}.$$

✓.

(C1)

(C2) By continuity and coercivity we have for some constants $C, \beta$

$$\beta \|u\|_H^2 \leq a(u,u) \leq C \|u\|_H^2 \quad (*)$$

hence, equivalence of norms and topologies. We have to check

(i) $\|u\|_a = \sqrt{a(u,u)}$ is a norm

(ii) $a(u,v)$ is a scalar product on $H$.

It will be helpful first to prove Cauchy-Schwartz inequality for $a(u,v)$, i.e $|a(u,v)|^2 \leq a(u,u)\,a(v,v) \quad \forall u,v \in H$.

Proof: let $\lambda \in \mathbb{R}$. We compute

$$0 \leq a(u-\lambda v, u-\lambda v) = a(u,u) - 2\lambda a(u,v) + \lambda^2 a(v,v)$$

Let $\lambda = \frac{a(u,v)}{a(v,v)}$. Then $0 \leq a(u,u) - 2\frac{a(u,v)^2}{a(v,v)} + \frac{a^2(u,v)}{a(v,v)}$

$$\Rightarrow \quad a(u,v)^2 \leq a(u,u)\,a(v,v).$$

From this we prove that $\|u\|_a$ satisfies triangle inequality.

$$\|u+v\|_a^2 = \|u\|_a^2 + \|v\|_a^2 + 2a(u,v) \leq$$

$$\leq \|u\|_a^2 + \|v\|_a^2 + 2\|u\|_a \|v\|_a = (\|u\|_a + \|v\|_a)^2$$

and the assertion follows.

Now, (i) and (ii) are trivial as $a$ is bilinear symmetric and satisfies $(*)$.

$\square$.

3 We apply 2. Now, originally $\ell$ is bounded linear functional on $(H, \langle\cdot;\cdot\rangle_H)$ but by equivalence of norms it is also a bounded linear functional on $(H, \langle\cdot;\cdot\rangle_A)$ (indeed, $\exists_C$ $|\ell(v)| \leq C\|v\|_H \Rightarrow |\ell(v)| \leq \tilde{C}\|v\|_A$).

By Riesz Representation Theorem, there is $\tilde{\ell}$ s.t. $\ell(v) = a(\tilde{\ell}, v)$ for all $v \in H$. Therefore, problem to be solved reads:

$$\forall_{v \in H} \quad a(u - \tilde{\ell}, v) = 0$$

Hence $u - \tilde{\ell} \in H^{\perp} = \{0\}$ and so $u = \tilde{\ell}$. By this, such $u$ is unique

orthogonal complement wrt a

By equivalence of norms, $u$ belongs to the original space.

$*$: in special problems PDEs to be solved with Lax-Milgram.