Functional Analysis, PS 7

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A1

In Hilbert spaces: $x_n \rightharpoonup x$ iff

$$\forall_{u \in H} \quad \langle x_n, u \rangle \longrightarrow \langle x, u \rangle.$$

In L^p $(1 \leq p < \infty)$ $f_n \rightharpoonup f$ iff $\forall_{g \in L^{p'}}$

$$\int f_n g \longrightarrow \int fg.$$

A2

$$\begin{matrix} x_n \rightharpoonup x \\ y_n \rightharpoonup y \end{matrix} \Rightarrow \forall_{\varphi \in E^*} \varphi(x) = \varphi(y) \Rightarrow x = y.$$

A3

$$x_n \rightharpoonup x \Rightarrow \forall_{\varphi} \quad \varphi(x_n) \text{ is convergent} \Rightarrow$$

$$\Rightarrow \forall_{\varphi} \quad \{\varphi(x_n)\} \text{ is bdd} \underset{\text{B-S}}{\Rightarrow} \|x_n\| \leq C.$$

$$\|x\| = \sup_{\|\varphi\| \leq 1} |\varphi(x)| = \sup_{\|\varphi\| \leq 1} \lim_{n \to \infty} |\varphi(x_n)| \leq$$

$$\leq \liminf_{n \to \infty} \sup_{\|\varphi\| \leq 1} |\varphi(x_n)| \leq \liminf_{n \to \infty} \|x_n\|.$$

A4 When $x_n \to x$, $\forall_{\varphi \in E^*}$

$$|\varphi(x_n) - \varphi(x)| = |\varphi(x_n - x)| \leq \|\varphi\| \, \|x_n - x\| \to 0.$$

A5 $x_n \rightharpoonup x$: $\forall_{f \in E^*}$ $f(x_n) \to f(x)$

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$$

$$\leq \underbrace{\|f_n - f\|}_{\to 0} \underbrace{\|x_n\|}_{\text{bdd}} + \underbrace{|f(x_n) - f(x)|}_{\to 0} \to 0.$$

A6 [Mazur $\Rightarrow$ $\Uparrow$].

B1 We want: $\forall_{y \in l^2} \quad y_i \longrightarrow 0$ as $i \to \infty$

But this is easy as $\sum |y_i|^2 < \infty$.

It follows that $e_n \rightharpoonup 0$. But $e_n \not\to 0$. Indeed,

$$\|e_n\|_{l^2} = 1 \quad \forall_n \quad (\text{i.e. } \|e_n - 0\| = 1).$$

$\square$.

B2 $\sin(nx) \rightharpoonup 0$ in $L^2(0, 2\pi)$

We want $\displaystyle\int_0^{2\pi} f(x) \sin(nx) \to 0 \quad \forall_{f \in L^2(0,2\pi)}$.

First, we prove this for $f = \sum_{i=1}^{N} a_i \mathbb{1}_{(b_i, b_{i+1}]}$

$$\int_0^{2\pi} f(x) \sin(nx)\, dx = \sum_{i=1}^{N} a_i \int_{b_i}^{b_{i+1}} \sin(nx)\, dx =$$

$$= \frac{1}{n} \sum_{i=1}^{N} a_i \left[\cos(n b_{i+1}) - \cos(n b_i)\right] \to 0$$

as $n \to \infty$.

Now, given arbitrary $f \in L^2(0, 2\pi)$, there exists sequence f_m of simple functions $|f_m| \le |f|$ and $f_n \longrightarrow f$ a.e. on $(0, 2\pi)$. It follows, by Dominated convergence $f_n \longrightarrow f$ in $L^2(0, 2\pi)$.

Fix $\varepsilon > 0$. Choose f_m s.t. $\|f_m - f\|_2 \le \varepsilon$.

then

$$\left| \int f(x) \sin(nx)\, dx \right| \le \left| \int (f_m(x) - f(x)) \sin(nx)\, dx \right|$$

$$+ \left| \int f_m(x) \sin(nx)\, dx \right| \le$$

$$\le \underbrace{\|f_m - f\|_2}_{\le \varepsilon} \underbrace{\|\sin(nx)\|_2}_{\text{bdd}} + \underbrace{\left| \int f_m \sin(nx)\, dx \right|}_{\to 0 \text{ as } n \to \infty}$$

$$\Rightarrow \limsup_{n \to \infty} \left| \int f(x) \sin(nx)\, dx \right| \le C\varepsilon$$

$$\Rightarrow \limsup_{n \to \infty} \left| \int f(x) \sin(nx)\, dx \right| = 0.$$

Btw, we proved that simple functions are dense in $L^p(\Omega)$ $(1 \le p < \infty)$.

$\Uparrow$: $\sin^2(nx) \rightharpoonup 1/2$

(B3) Convergence of $\sin(nx)$: weak, strong and a.e.

We know $\sin(nx) \rightharpoonup 0$ in $L^2(0,2\pi)$.

If $\{\sin(nx)\}$ was convergent in $L^2(0,2\pi)$, we would have $\sin(nx) \to 0$ in $L^2(0,2\pi)$.

But

$$\|\sin(nx)\|_2^2 = \int_0^{2\pi} \sin^2(nx)\,dx \underset{y=nx}{=} \frac{1}{n}\int_0^{2\pi n} \sin^2(y)\,dy$$

$$= \int_0^{2\pi} \sin^2(y)\,dy > 0.$$

Finally, if $\sin(nx) \to 0$ a.e., by Dominated Convergence, $\sin(nx) \to 0$ in $L^2(0,2\pi)$.

$\square$.

B4

$x_n \rightharpoonup x$

$$\limsup_{n\to\infty} \|x_n\|^2 \le \|x\|^2.$$

$$\|x_n - x\|^2 = \langle x_n - x, x_n - x \rangle =$$

$$= \langle x_n, x_n \rangle - \langle x, x_n \rangle - \langle x_n, x \rangle + \langle x, x \rangle$$

$$= \|x_n\|^2 - 2\langle x, x_n \rangle + \|x\|^2$$

$$\limsup_{n\to\infty} \|x_n - x\|^2 \le \limsup_{n\to\infty} \|x_n\|^2 +$$

$$+ \limsup_{n\to\infty} \Big[\|x\|^2 - 2\overbrace{\langle x, x_n \rangle}^{\to \|x\|^2} \Big]$$

$$\longrightarrow -\|x\|^2$$

$$\le \|x\|^2 - \|x\|^2 = 0.$$

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