Functional Analysis (WS 20/21)

additional basic problems

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1. Consider

$$X = \left\{ f \in C(\mathbb{R}) : \lim_{x \to \pm\infty} x^2 \, |f(x)| = 0 \right\}, \qquad \|f\|_X = \sup_{x \in \mathbb{R}} x^2 \, |f(x)|.$$

   Is $(X, \|\cdot\|_X)$ a normed space? Is it a Banach space?

2. Consider $Y = C[0,1]$ and $\|f\|_Y = \sum_{k=1}^{\infty} \frac{1}{k^2} \, f\left(\frac{1}{k}\right)$. Is $(Y, \|\cdot\|_Y)$ a normed space? Is it a Banach space?

3. Consider

$$X = \left\{ f \in C^1(\mathbb{R}) : \int_{-\infty}^{+\infty} |f'(x)| \, dx < \infty \right\}$$

   where $C^1(\mathbb{R})$ is the space of continuously differentiable functions on $\mathbb{R}$. Prove that X equipped with a norm

$$\|f\| = |f(0)| + \int_{-\infty}^{+\infty} |f'(x)| \, dx$$

   is a normed space. Is it a Banach space?

4. Let $T : l^1 \to c_0$ be defined with

$$(Tx)_n = \sum_{k=n}^{\infty} x_k.$$

   Prove that T is well-defined (i.e. for $x \in l^1$, we have $Tx \in c_0$) and bounded linear operator. Compute its norm.

5. For $f : \mathbb{R}^+ \to \mathbb{R}^+$ we define

$$\varphi(f) = \int_{\mathbb{R}^+} f(t) \, e^{-t} \, \mathrm{d}t.$$

   Find all p $(1 \leq p \leq \infty)$ such that $\varphi \in (L^p(\mathbb{R}^+))^*$? For such p compute norm of φ as a functional on $L^p(\mathbb{R}^+)$.