Functional Analysis (WS 20/21)

Homeworks

Compiled on 21/01/2021 at 10:51am

General instruction: Problems have to be solved in groups of 2 students and the solutions have to be e-mailed to jakub.skrzeczkowski@student.uw.edu.pl before the class begins (12:15). As the subject of your e-mail use

AF-**n**-homework

where **n** is the number of the submitted homework (**n** is 1 for the first one).

Contents

Homework 1: problems for 22/10/2020

1. (**Littlewood's interpolation inequality**) Let $f \in L^p \cap L^q$ for some $1 \leq p, q \leq \infty$. Prove that for $r \in [p, q]$ we have $f \in L^r$ and $\|f\|_r \leq \|f\|_p^\alpha \|f\|_q^{1-\alpha}$ for some $\alpha \in [0,1]$. *Hint:* let $\alpha \in [0,1]$ and write $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$.

2. Let $\|f\|_B := |f(0)| + \sup_{x \in [0,1]} |f'(x)|$. Prove that $(C^1[0,1], \|\cdot\|_B)$ is a normed space. Is it a Banach space?

Homework 2: problems for 29/10/2020

1. For $\alpha \in (0,1)$ we define space $C^\alpha[0,1]$ of Hölder continuous functions with exponent α, i.e. of functions $f \in C[0,1]$ such that

$$|f|_\alpha := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty.$$

In this problem we take (for simplicity) $\alpha = \frac{1}{2}$. Are the following pairs normed spaces? Are they Banach spaces? Justify your answer.

   a. $(C^{1/2}[0,1], |\cdot|_{LIP})$
   b. $(C^{1/2}[0,1], |\cdot|_{1/2})$
   c. $(C^{1/2}[0,1], \|\cdot\|_\infty + |\cdot|_{LIP})$
   d. $(C^{1/2}[0,1], \|\cdot\|_\infty + |\cdot|_{1/2})$
   e. $(C^{1/2}[0,1], \|\cdot\|_\infty + |\cdot|_{LIP} + |\cdot|_{1/2})$

Hint: What is a natural example of a function being in $C^{1/2}[0,1]$?

2. a. For $1 \leq p < \infty$ we define l^p as the space of sequences $x = (x_1, x_2, ...)$ summable with p-th power and equipped with the norm

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$$

For $p = \infty$, we define l^∞ as bounded sequences $x = (x_1, x_2, ...)$ with the norm

$$\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k|.$$

Use the theory presented in the lectures to justify briefly that $(l^p, \|\cdot\|_p)$ is a Banach space (for $1 \leq p \leq \infty$). More precisely, we know that for any σ-finite space $(X, \mathcal{F}, \mu)$ and $1 \leq p \leq \infty$, $L^p(X, \mathcal{F}, \mu)$ with a usual L^p norm is a Banach space. Choose appropriate X and μ.

   b. Consider functional $\varphi : l^1 \to \mathbb{R}$ defined with $\varphi(u) = \sum_{i=1}^{\infty} \frac{1}{2^i} u_i$ where $u = (u_1, u_2, u_3, ...)$. Prove that $\varphi \in (l^1)^* = \mathcal{L}(l^1, \mathbb{R})$ and compute its norm $\|\varphi\|$.

Homework 3: problems for 5/11/2020

1. Let $1 \leq p \leq \infty$. Compute norms of the operators

   a. $T : l^p \to l^p$ defined with $T\left((a_n)_{n\geq 1}\right) = (a_{n+1} - a_n)_{n\geq 1}$.

   b. $T : L^p(0,1) \to L^p(0,1)$ defined with $(Tf)(x) = f(\sqrt{x})$.

2. Let $(X, \|\cdot\|_X)$ be a normed space and $(Y, \|\cdot\|_Y)$ be a Banach space. Suppose that D is a dense linear subspace of X and $T : (D, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is a bounded linear operator. Prove that T has a <u>unique</u> bounded extension

   $$\widetilde{T} : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$$

   such that
   $$Tx = \widetilde{T}x \text{ for } x \in D \qquad \text{and} \qquad \|T\|_{\mathcal{L}(D,Y)} = \|\widetilde{T}\|_{\mathcal{L}(X,Y)}.$$

   Hint: If $x \in X \setminus D$, there is a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ such that $\|x_n - x\|_X \to 0$ as $n \to \infty$.
   Hint 2: So you have to prove that the extension exists and that it is unique.

Homework 4: problems for 12/11/2020

1. Let F be a normed space $C[0,1]$ with L^2 norm, i.e. $F = (C[0,1], \|\cdot\|_2)$. For $n \in \mathbb{N}$, we define

$$\varphi_n(f) = n \int_0^{\frac{1}{n}} f(t)\, dt.$$

   Verify that:

   - $\varphi_n : F \to \mathbb{R}$ is a bounded linear functional on F, i.e. $\varphi \in F^*$,
   - for every fixed $f \in F$, we have $\sup_{n\in\mathbb{N}} |\varphi_n(f)| < \infty$,
   - we have $\sup_{n\in\mathbb{N}} \|\varphi_n\| = \infty$.

   Why Banach-Steinhaus Theorem is not satisfied in this case?

2. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces. Let $a : E \times F \to \mathbb{R}$ be a bilinear form such that

   - for fixed $x \in E$, the map $F \ni y \mapsto a(x,y)$ is continuous (so it belongs to F^*), i.e. for each $x \in E$, there is a constant C_x (it may depend on x) such that

$$|a(x,y)| \leq C_x \|y\|_F.$$

   - for fixed $y \in F$, the map $E \ni x \mapsto a(x,y)$ is continuous (so it belongs to E^*), i.e. for each $y \in F$, there is a constant C_y (it may depend on y) such that

$$|a(x,y)| \leq C_y \|x\|_E.$$

   Prove that there is a constant C (independent of x and y) such that

$$|a(x,y)| \leq C \|x\|_E \|y\|_F$$

   for all $x \in E$ and $y \in F$.

   Remark: Roughly speaking, separately continuous linear maps on Banach spaces are actually jointly continuous.

Homework 5: problems for 19/11/2020

1. **(Minty monotonicity trick)** Let $(X, \|\cdot\|_X)$ be a Banach space. Consider a linear operator $T : X \to X^*$ such that for all $x \in X$:

$$(Tx)(x) \geq 0.$$

   Prove that T is a bounded linear operator, i.e. $T \in \mathcal{L}(X, X^*)$.

   Hint: For sequence $x_n \to x$ in X and arbitrary $z \in X$, consider $x - x_n + \varepsilon\, z$ where $\varepsilon > 0$ is arbitrarily small.

2. We write c_0 for the space of sequences $x = (x_1, x_2, ...)$ such that $\lim_{n\to\infty} x_n = 0$ (i.e. sequences converging to 0). Space c_0 is equipped with the usual supremum norm $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.

   - Prove that $c_0 \subset l^\infty$.
   - Prove that $(c_0, \|\cdot\|_\infty)$ is a Banach space.
   - Let $z = (z_1, z_2, ...)$ be a sequence of real numbers such that whenever $y = (y_1, y_2, ...) \in c_0$, we have that $\sum_{n\geq 1} z_n\, y_n$ is convergent in $\mathbb{R}$. Prove that $\sum_{n\geq 1} |z_n|$ is convergent.

     Hint: for $y \in c_0$, consider $\varphi_k \in (c_0)^*$ defined with $\varphi_k(y) = \sum_{n=1}^{k} z_n\, y_n$.

Homework 6: problems for 26/11/2020

1. Let $1 \leq p < \infty$. Prove that $L^p(0,1)$ equipped with $L^q(0,1)$ norm for $1 \leq q < p$ is not a Banach space.

2. Let $(X, \|\cdot\|_X)$ be a Banach space and $P : X \to X$ be a linear operator such that

   - the kernel of P is closed in X,
   - the image of P is closed in X,
   - $P(P(x)) = P(x)$ for all $x \in X$.

   Prove that $P \in \mathcal{L}(X, X)$.

3. Consider $L^2(-1,1)$ as a linear space over $\mathbb{R}$. Let

$$E = \left\{ f \in L^2(-1,1) : \int_{-1}^{1} f(t)\,\mathrm{d}t = \int_{-1}^{1} f(t)\,t\,\mathrm{d}t = 0 \right\}.$$

   Let $g(x) = \frac{1}{1+x^2}$. Find $P_E\, g$, $P_{E^\perp}\, g$ and distance of g from E.

Homework 7: problems for 3/12/2020

1. Let H be a Hilbert space, $\varphi \in H^*$ and $T \in \mathcal{L}(H, H)$. Prove or disprove: there exists uniquely determined $y \in H$ such that for all $x \in H$ we have $\varphi(Tx) = \langle x, y \rangle$.

Homework 8: problems for 10/12/2020

1. Prove that there is a bounded functional on l^∞ denoted with φ (called Banach limit) such that

   - $\varphi((x_0, x_1, x_2, ...)) = \varphi((x_1, x_2, x_3, ...))$, i.e. φ does not depend on finitely many terms,
   - for $x \in l^\infty$ we have $\liminf_{n\to\infty} x_n \le \varphi(x) \le \limsup_{n\to\infty} x_n$,
   - for converging $x \in l^\infty$ we have $\varphi(x) = \lim_{n\to\infty} x_n$.

   Hint: consider subspace $W = \{x \in l^\infty : \lim_{n\to\infty} \frac{x_1+x_2+...+x_n}{n} \text{ exists}\}$. Observe that when $x_n \to \alpha$, we also have $\frac{x_1+x_2+...+x_n}{n} \to \alpha$.

2. **(Mazur Lemma)** Let $C \subset E$ be a convex set. Prove that C is closed for convergence in norm if and only if C is closed for weak convergence. *Hint*: One implication is trivial, for another use geometric version Hahn-Banach.

   Definitions:
   <u>C is closed for convergence in norm</u> if for any $\{x_n\}_{n\ge1} \subset C$ such that $x_n \to x$ it follows that $x \in C$. This is exactly the same as statement that C is closed in E.
   <u>C is closed for weak convergence</u> if for any $\{x_n\}_{n\ge1} \subset C$ such that $x_n \rightharpoonup x$ it follows that $x \in C$.

Homework 9: problems for 17/12/2020

1. Let $I : L^2(0,1) \to \mathbb{R}$ be a (nonlinear!) function defined with

$$I(u) = \int_0^1 |u(x)|^{3/2} \cos^2(x)\, \mathrm{d}x.$$

   (A) Prove that I is continuous on $L^2(0,1)$.

   (B) Prove that the set
   $$\{(u,\lambda) \in L^2(0,1) \times \mathbb{R} : I(u) < \lambda\}$$
   is open and convex.

   (C) Fix $u \in L^2(0,1)$. Prove that there exists $v_u \in L^2(0,1)$ such that for all $w \in L^2(0,1)$ we have
   $$I(u+w) \geq I(u) + \langle w, v_u \rangle.$$
   What is v_u in the language of classical calculus for convex functions?

   Remark: It is quite instructive to see why the version of Hahn-Banach theorem with compact/closed sets cannot be used here.

2. (A) Let $f : (0,1) \to \mathbb{R}$ be a bounded function and extend it periodically to the function $f : \mathbb{R} \to \mathbb{R}$. Prove that $f_n(x) = f(n\,x)$ converges weakly in $L^2(0,1)$ to $\overline{f} = \int_0^1 f(y)\, \mathrm{d}y$ (i.e. constant function).

   (B) Find weak limits in $L^2(0,1)$ of sequences $\sin(2\pi n x)$ and $\sin^2(2\pi n x)$.

Homework 10: problems for 7/01/2021

1. Let H be a Hilbert space. Below are some simple exercises on orthonormal sets and basis.

   (A) Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal set in Hilbert space H. Consider operator $T : H \to c_0$ defined with
   $$Tx = \left(\frac{n}{n+1}\langle x, e_n\rangle\right)_{n\in\mathbb{N}}.$$
   Prove that T is well-defined. Is T a bounded linear operator? If yes, compute its norm.

   (B) Let H be an infinite dimensional Hilbert space. Prove that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\|x_n\| = 1$ and $x_n \rightharpoonup 0$.

   (C) Let $y \in l^\infty$, $\{e_n\}_{n\in\mathbb{N}}$ is an orthonomal set in H and $u_n = \frac{1}{n}\sum_{i=1}^n e_i\, y_i$. Prove that $u_n \to 0$ strongly in H.

   (D) Let $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ be an orthonormal basis of H. Use density argument to prove that
   $$x_n \rightharpoonup x \text{ in } H \iff \langle x_n - x, e_\alpha\rangle \to 0 \text{ for all } \alpha \in \mathcal{A} \text{ and } \{x_n\}_{n\in\mathbb{N}} \text{ is bounded in } H^{1}.$$

2. Let μ be a gaussian measure on $\mathbb{R}$, i.e. a measure with density $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Let $H = L^2(\mathbb{R}, \mu)$.

   (A) Let $X = \mathrm{span}(1, x, x^2)$. Prove that X is a linear subspace of H.

   (B) Recall Gram-Schmidt algorithm. Use it to find an orthonormal basis of X.

   (C) Compute the distance of $f(x) = |x|$ from X.

   When evaluating integrals you may use certain moments of normally distributed random variables without a proof.

[1] Correction: 29/12/2020

Homework 11: problems for 14/01/2021

1. The following exercise shows that any compact and nonempty $K \subset \mathbb{C}$ is a spectrum of some $T : l^2 \to l^2$.

   (A) Let $y \in l^\infty$ be a complex sequence and $T : l^2 \to l^2$ be defined with $Tx = (y_i\, x_i)_{i\in\mathbb{N}}$. Prove that $\sigma(T) = \overline{\{y_i : i \in \mathbb{N}\}}$ where the line above denotes closure of the set.

   (B) Let $K \subset \mathbb{C}$ be a nonempty and compact subset. Construct a bounded linear operator $T : l^2 \to l^2$ such that $\sigma(T) = K$.

2. For the following operators briefly justify that they are bounded and linear. Find their adjoints.

   (A) $K : L^2(0,1) \to L^2(0,1)$ defined with $Kf(x) = \int_0^x f(y)\,\mathrm{d}y$.

   (B) $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined with $Tf(x) = \mathrm{sgn}(x) f(x+1)$.

Homework 12: problems for 21/01/2021

1. Consider the right and left shifts operators on $l^2(\mathbb{N})$ (we usually denote this space with l^2) defined with
$$Rx = (0, x_1, x_2, ...), \qquad Lx = (x_2, x_3, x_4, ...).$$
Find $\sigma(R)$ and $\sigma(L)$. Are these operators compact on $l^2(\mathbb{N})$?

2. For the following operators decide whether they are compact or not.

   (A) Identity operator
   $$I : L^p(0,1) \to L^q(0,1), \qquad p \geq q \geq 1,$$
   Suggestion: Consider oscillating sequence $f_n(x) = \sin(2\pi nx)$ and apply dominated convergence.

   (B) Identity operator
   $$I : C^{1/2}[0,1] \to C[0,1].$$
   Here, $C^{1/2}[0,1]$ is the Banach space equipped with norm $\|\cdot\|_\infty + |\cdot|_{1/2}$ cf. Problem 1 in Homework 2. *Suggestion:* Arzela-Ascoli.

Homework 13: problems for 28/01/2021

1. In what follows we find all complex-valued sequences $y \in l^\infty$ such that the multiplication operator $T : l^2 \to l^2$ defined with $Tx = (y_i\, x_i)_{i\in\mathbb{N}}$ is compact.

   Let H be a Hilbert space and $\mathcal{K}(H,H)$ be a space of compact operators equipped with operator norm (so this is a subspace of $\mathcal{L}(H,H)$). Prove the following properties of $\mathcal{K}(H,H)$:

   (A) $\mathcal{K}(H,H)$ is closed in $\mathcal{L}(H,H)$ so it is itself a Banach space.

   (B) If $K \in \mathcal{K}(H,H)$ and $S \in \mathcal{L}(H,H)$ then $KS, SK \in \mathcal{K}(H,H)$.

   (C) If image of $T \in \mathcal{L}(H,H)$ is finite dimensional, $T \in \mathcal{K}(H,H)$.

   Use (A), (C) and spectral theory of compact operators to prove that the multiplication operator T is compact if and only if $y \in c_0$. Compare with multiplication operator on $L^2(\mathbb{R})$.

2. Here are some simple exercises on mollifiers, density arguments etc.

   (A) Let $f \in L^1(\Omega)$. Prove that if $\int_\Omega f\,\varphi \geq 0$ for all $\varphi \in C_c^\infty(\Omega), \varphi \geq 0$ then $f \geq 0$ a.e. in Ω.

   (B) Let Q be a d-dimensional cube with side length 1. Prove that there is a smooth function f such that $f = 1$ on Q, $f = 0$ on $\mathbb{R}^n \setminus 2Q$ and $f \in [0,1]$.[2]

   (C) Write $\mathbb{R}^d = \cup_{i=1}^\infty Q_i$ where Q_i are d-dimensional disjoint cubes with side length 1. Prove that there exists f_i such that f_i is supported on $2\,Q_i$, $f_i \in [0,1]$ and $\sum_{i=1}^\infty f_i = 1$[3].[4]

   (D) Let $1 < p < \infty$. Prove that $f_n \rightharpoonup f$ in $L^p(\Omega)$ if and only if

$$\int_\Omega f_n\,\varphi \to \int_\Omega f\,\varphi \text{ for all } \varphi \in C_c^\infty(\Omega) \text{ and } \{f_n\}_{n\in\mathbb{N}} \text{ is bounded in } L^p(\Omega).$$

[2] We say that f is a cutoff of Q. Note that given $\varepsilon > 0$ we can in fact construct f_ε such that f_ε is supported on $(1+\varepsilon)\,Q$. Such cutoffs are extremely useful in PDEs: they allow to localize problems around some small sets and as they are smooth - they interplay nicely with derivatives contrary to the simple characteristic functions.

[3] This series is convergent in the following sense: for each $x \in \mathbb{R}^d$ only finitely many terms may be nonzero.

[4] The family $\{f_i\}_{i\in\mathbb{N}}$ is called a smooth partition of unity. In applications, one writes $u = u \cdot 1 = \sum_{i=1}^\infty u\, f_i$. Now, $u\, f_i$ is located on some smaller set. Usually this argument is performed when we work close to the smooth boundary γ of some set being a curve. In a small neighbourhood of $x \in \gamma$ it can be assumed to be a graph of some function, hence we localize around this neighbourhood and the boundary is given explicitly by some function.