Functional Analysis (WS 20/21), Problem Set 10

(adjoint and self-adjoint operators on Hilbert spaces)[1]

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In what follows, let H be a **complex** Hilbert space.

Let $T : H \to H$ be a bounded linear operator. We write $T^* : H \to H$ for **adjoint** of T defined with

$$\langle Tx, y\rangle = \langle x, T^* y\rangle.$$

This operator exists and is uniquely determined by Riesz Representation Theorem.

Basic facts on adjoint operators

Properties A1, A2, A3, A7, A9 were discussed in the lecture.

A1. Adjoint T^* exists and is uniquely determined.

A2. Adjoint T^* is a bounded linear operator and $\|T^*\| = \|T\|$.

A3. Taking adjoints is an involution: $(T^*)^* = T$.

A4. Adjoints commute with the sum: $(T_1 + T_2)^* = T_1^* + T_2^*$.

A5. For $\lambda \in \mathbb{C}$ we have $(\lambda T)^* = \overline{\lambda}\, T^*$.

A6. Let T be a bounded invertible operator. Then, $(T^*)^{-1} = (T^{-1})^*$.

A7. Let T_1, T_2 be bounded operators. Then, $(T_1 T_2)^* = T_2^* T_1^*$.

A8. We have relationship between kernel and image of T and T^*:

$$\text{ker } T^* = (\text{im } T)^\perp, \qquad (\text{ker } T^*)^\perp = \overline{\text{im } T}$$

It will be helpful to recall that if $M \subset H$ is a linear subspace, then $\overline{M} = \left(M^\perp\right)^\perp$.

A9. Spectrum $\sigma(A^*) = \{\lambda \in \mathbb{C} : \overline{\lambda} \in \sigma(A)\}$.

Computation of adjoints

B1. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a complex matrix. Find A^*.

B2. Let $H = l^2(\mathbb{Z})$. For $x = (..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...) \in H$ we define the right shift operator with $(Rx)_k = x_{k-1}$. Find $\|R\|$, R^{-1} and R^*. Similarly, one can consider the left shift operator L. Find $\|L\|$, L^{-1} and L^*

B3. Let $K : L^2(0,1) \to L^2(0,1)$ be defined with $Kf(x) = \int_0^x f(y)$. Prove that K is a bounded linear operator and compute K^*.

B4. Let $M \subset H$ be a closed subspace and P_M be an orthogonal projection on M. Find $(P_M)^*$.

[1] A useful reference for this topic is Chapter 9 of the book *Applied Analysis* by John Hunter and Bruno Nachtergaele available online at https://www.math.ucdavis.edu/ hunter/book/pdfbook.html. It may be helpful to read Wikipedia articles: "Hermitian adjoint"'.

B5. Let $A : H \to H$ be a bounded operator. Recall that e^A exists as a series $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ converging in the operator norm. Compute $\left(e^A\right)^*$.

B6. Let $T : L^2(0,1) \to L^2(0,1)$ be defined with

$$Tf(x) = \int_0^1 k(x,y)f(y)dy$$

for some bounded and measurable function $k(x,y)$. Find the adjoint of T. *Remark:* This operator is called Hilbert-Schmidt operator.

B7. Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined with $Tf(x) = \text{sgn}(x)f(x+1)$. Prove that T is well - defined and find T^*.

Self-adjoint operators

The following properties were discussed in the lecture:

- If $T : H \to H$ is self-adjoint then $\sigma(T)$ is real.
- If $T : H \to H$ is self-adjoint then its eigenvectors corresponding to different eigenvalues are orthogonal.

C1. Prove that if $T : H \to H$ satisfies $\langle Tx, y\rangle = \langle x, Ty\rangle$ then T is bounded.

C2. Prove that $T : H \to H$ is self-adjoint if and only if $\langle Tx, x\rangle$ is real for all $x \in H$.

C3. Let $M \subset H$ be a closed subspace. Recall what is the adjoint of the orthogonal projection on M denoted with P_M? What is $\sigma(P_M)$ and what are components of this spectrum (point, continuous, residual)?

C4. Let $M : L^2(0,1) \to L^2(0,1)$ be a multiplication operator defined with $Mf(x) = xf(x)$. Prove that M is self-adjoint. Recall what is the spectrum of M.

C5. More generally, let g be a bounded, continuous function and consider multiplication operator $G : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined with $Gf(x) = g(x)f(x)$. Recall what is the spectrum of G. Find sufficient and necessary condition on g so that G is a self-adjoint operator.