Functional Analysis (WS 20/21), Problem Set 13

(Fourier transform)

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For $f \in L^1(\mathbb{R}^n)$ we define Fourier transform of f with

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, \mathrm{d}x.$$

Fourier transform is a continuous isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. The inverse is given with

$$\check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \xi \cdot x} \, d\xi.$$

According to Plancherel theorem if $f, g \in \mathcal{S}(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)} \, dx = \int_{\mathbb{R}^n} \widehat{f(x)} \, \overline{\widehat{g(x)}} \, dx$$

so that if $f = g$ we see that Fourier transform extends to $L^2(\mathbb{R}^n)$ by density argument and is an isometrical isomorphism from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

1. For $f \in L^1$, $\|\hat{f}\| \le \|f\|_1$, $\hat{f}$ is continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. This is Riemman-Lebesgue Lemma.
2. Under Fourier transform:
   (A) convolution becomes multiplication: $\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.
   (B) translation becomes rotation: $\widehat{\tau_h f}(\xi) = \hat{f}(\xi) e^{2\pi i \xi \cdot h}$ where $\tau_h f(x) = f(x+h)$.
   (C) differentiation becomes multiplication by a polynomial: $\widehat{f_{x_j}}(\xi) = 2\pi i \xi \hat{f}(\xi)$.
   (D) find $\widehat{\delta_h f}$ where $\delta_h f(x) = f(x/h)$.
3. Compute $\hat{f}$ for $f(x) = e^{-\pi |x|^2}$ and $f(x) = e^{-x}\chi_{[0,\infty)}(x)$ (in 1D).
4. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Find all $u \in \mathcal{S}(\mathbb{R}^n)$ such that $-\Delta u - u = f$ in $\mathbb{R}^n$.
5. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Find all $u \in \mathcal{S}(\mathbb{R}^3)$ such that $f = u + \partial_1^2 \partial_2^2 \partial_3^4 u + 4i\partial_1^2 u + \partial_2^7 u$.
6. (**Heisenberg uncertainity principle**) Let $\psi \in \mathcal{S}(\mathbb{R})$ with $\|\psi\|_2 = 1$. Prove that
   $$\left[\int_{\mathbb{R}} x^2 |\psi(x)| \, dx\right] \cdot \left[\int_{\mathbb{R}} \xi^2 |\hat{\psi}(\xi)|^2 \, d\xi\right] \ge \frac{1}{16\pi^2}.$$
7. Let $g \in L^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$.
   (A) Let $M : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be given with $Mf = \hat{g} f$. Prove that M is well-defined, i.e. it has image in $L^2(\mathbb{R})$.
   (B) Prove that $\sigma(M) = \overline{\{\hat{g}(x) : x \in \mathbb{R}\}} = \{\hat{g}(x) : x \in \mathbb{R}\}$.
   (C) Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined with $Tf = f * g$. Prove that T is well-defined.
   (D) Find $\sigma(T)$.