Functional Analysis (WS 20/21), Problem Set 3

(Banach-Steinhaus Theorem)

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Baire Category Theorem: Let (X, d) be a complete metric space. Suppose that $K_i \subset X$ are closed and have empty interior. Then $\bigcup_{i=1}^{\infty} K_i$ has empty interior.

Banach-Steinhaus Theorem: Let $(X, \|\cdot\|_X)$ be a Banach space and $(Y, \|\cdot\|_Y)$ be a normed space. Let $\{T_\alpha\}_{\alpha \in A}$ be a family of bounded linear operators between X and Y. Suppose that for any $x \in X$,

$$\sup_{\alpha \in A} \|T_\alpha x\|_Y < \infty.$$

Then $\sup_{\alpha \in A} \|T_\alpha\| < \infty$.

Baire Category Theorem

A1. Let $(X, \|\cdot\|_X)$ be an infinite dimensional Banach space. Prove that X has uncountable Hamel basis.

A2. Consider subset of bounded sequences

$$A = \{x \in l^\infty : \text{ only finitely many } x_k \text{ are nonzero}\}.$$

Can one define a norm on A so that it becomes a Banach space?

A3. Does there exist a norm $\|\cdot\|$ such that $(\mathcal{P}[0,1], \|\cdot\|)$ (space of polynomials on $[0,1]$) is a Banach space?

A4. Prove that the set $L^2(0,1)$ has empty interior as the subset of Banach space $L^1(0,1)$.

Banach-Steinhaus Theorem

B1. Let F be a normed space $C[0,1]$ with $L^2(0,1)$ norm, i.e. $F = (C[0,1], \|\cdot\|_2)$. For $n \in \mathbb{N}$, we define

$$\varphi_n(f) = n \int_0^{\frac{1}{n}} f(t)\, dt.$$

Verify that:

- φ_n defines a bounded linear functional on F, i.e. $\varphi \in F^*$,
- for every fixed $f \in F$, we have $\sup_{n \in \mathbb{N}} |\varphi_n(f)| < \infty$,
- we have $\sup_{n \in \mathbb{N}} \|\varphi_n\| = \infty$.

Why Banach-Steinhaus Theorem is not satisfied in this case?

B2. Let $(f_n)_{n \in \mathbb{N}} \subset L^2(0,1)$ be a sequence of functions in $L^2(0,1)$. Suppose that for each function $g \in L^2(0,1)$

$$\int_0^1 f_n(x) g(x) \mathrm{d}x \to C_g \in \mathbb{R}.$$

Prove that $\sup_{n \in \mathbb{N}} \|f_n\|_2 < \infty$.

B3. Let $(X, \|\cdot\|_X)$ be a Banach space and $A \subset X^*$ such that for every $x \in X$ the set

$$\{\varphi(x) : \varphi \in A\}$$

is bounded in $\mathbb{R}$. Prove that A is a bounded subset of X^*, i.e. $\sup\{\|\varphi\| : \varphi \in A\} < \infty$.

B4. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces. Let $a : E \times F \to \mathbb{R}$ be a bilinear form such that

- for fixed $x \in E$, the map $F \ni y \mapsto a(x, y)$ is continuous (so it belongs to F^*), i.e. for each $x \in E$, there is a constant C_x such that

$$|a(x, y)| \leq C_x \|y\|_F.$$

- for fixed $y \in F$, the map $E \ni x \mapsto a(x, y)$ is continuous (so it belongs to E^*), i.e. for each $y \in F$, there is a constant C_y such that

$$|a(x, y)| \leq C_y \|x\|_E.$$

Prove that there exists a constant C such that

$$|a(x, y)| \leq C \|x\|_E \|y\|_F$$

for all $x \in E$ and $y \in F$. Thus, linear maps that are separately continuous are actually jointly continuous. *Hint*: Problem B3. may be useful.

B5. Let X be the space of polynomials $\mathcal{P}[0, 1]$ equipped with the $L^1(0, 1)$ norm. We define a bilinear map: for $f, g \in X$, we put

$$\mathcal{B}(f, g) = \int_0^1 f(t)\, g(t)\, dt.$$

Check that $\mathcal{B}$ is separately continuous but it is not jointly continuous (in the sense of Problem B4.).

B6. Let $(x_n)_{n \geq 1}$ be a sequence of real numbers such that whenever $y = (y_1, y_2, ...) \in c_0$, we have that $\sum_{n \geq 1} x_n\, y_n$ is convergent. Prove that $\sum_{n \geq 1} |x_n|$ is convergent. *Hint*: for $y \in c_0$, consider $T_n \in (c_0)^*$ defined with $T_n(y) = \sum_{k=1}^{n} x_k y_k$.

B7. (**pointwise convergence of operators**) Let $(X, \|\cdot\|_X)$ be a Banach space and $(Y, \|\cdot\|_Y)$ be a normed space. Let $\{T_n\}_{n \in \mathbb{N}}$ be a family of bounded linear operators between X and Y such that for every $x \in X$, the sequence $T_n x$ converges to a limit denoted by Tx. Prove that

(a) $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$,

(b) T defines a bounded linear operator and $\|T\| \leq \liminf_{n \to \infty} \|T_n\|$.

B8. It is not true in general that pointwise limit of linear bounded operators defines a bounded operator. Here is an example. Let X be the space of sequences with only finitely many nonzero terms as in Problem B4.. Space X is equipped with usual supremum norm. For $x = (x_1, x_2, ...) \in X$, we define

$$T_n x = (x_1, 2\, x_2, ..., n\, x_n, 0, 0, ...)$$

so that $T_n : X \to X$. Prove that

(a) $T_n \in \mathcal{L}(X, X)$,

(b) for all $x \in X$, $\{T_n x\}_{n \in \mathbb{N}}$ converges in X and the limit defines operator $T : X \to X$,

(c) $T : X \to X$ is not bounded,

(d) $(X, \|\cdot\|_\infty)$ is not a Banach space.

B9. Let $(X, \|\cdot\|_X)$ be a Banach space and $A \subset X$ be its subset. Suppose that for every $\varphi \in X^*$, the set

$$\varphi(A) = \{\varphi(x) : x \in A\}$$

is bounded in $\mathbb{R}$. Prove that A is a bounded set in X (i.e. one can find a ball $B(0, R)$ for some $R > 0$ such that $A \subset B(0, R)$).