Functional Analysis (WS 20.21), Problem Set 6

(Dual spaces and Hahn-Banach theorems)

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Hahn-Banach Theorem (analytic form) Let $(X, \|\cdot\|)$ be a normed space and $M \subset X$ be a linear subspace. Let $p : X \to \mathbb{R}$ be such that

$$p(x+y) \leq p(x) + p(y), \qquad p(tx) = tp(x)$$

for all $x, y \in X$ and $t \geq 0$. Finally, suppose that $g : M \to \mathbb{R}$ is a linear functional and $g(x) \leq p(x)$ for all $x \in M$. Then, there exists a linear functional $f : X \to \mathbb{R}$ such that $f(x) = g(x)$ on M and $f(x) \leq p(x)$ for all $x \in X$.

See also Problem B1 for a simpler version of this result.

Hahn-Banach Theorem (geometric form) Let $(X, \|\cdot\|)$ be a normed space. Let $A, B \subset X$ be nonempty, convex and disjoint sets.

1. If A is open, there exists $\varphi \in X^*$ and λ such that

$$\varphi(x) < \lambda \leq \varphi(y)$$

   for all $x \in A$ and $y \in B$. We say that hyperplane $\{x \in X : \varphi(x) = \lambda\}$ separates A and B.

2. If A is closed and B is compact, there exists $\varphi \in X^*$ and λ_1, λ_2 such that

$$\varphi(x) < \lambda_1 < \lambda_2 < \varphi(y)$$

   for all $x \in A$ and $y \in B$. Let $\lambda = \frac{\lambda_1+\lambda_2}{2}$. We say that hyperplane $\{x \in X : \varphi(x) = \lambda\}$ separates strictly A and B.

Dual spaces characterization

A1. Let H be a Hilbert space. Recall from the lecture that $H = H^*$ in the sense of isometric isomorphism. Write explicitly this isomorphism.

A2. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Recall from the lecture that for $1 \leq p < \infty$, $(L^p)^* = L^q$ in the sense of isometric isomorphism (here $1/p + 1/q = 1$). Write explicitly this isomorphism.

A3. For $f : \mathbb{R}^+ \to \mathbb{R}^+$ we define

$$\varphi(f) = \int_{\mathbb{R}^+} f(t)\, e^{-t}\, \mathrm{d}t.$$

Find all p ($1 \leq p \leq \infty$) such that $\varphi \in (L^p(\mathbb{R}^+))^*$? For such p compute norm of φ as a functional on $L^p(\mathbb{R}^+)$.

A4. What is $(\mathbb{R}^n)^*$?

A5. Let X be a normed space. What is $(X \times \mathbb{R})^*$?

A6. Let $0 < p < 1$. Then, $L^p(0,1)$ can be still considered as a metric space equipped with metric

$$d_p(f,g) = \int_0^1 |f(x) - g(x)|^p \, dx.$$

Prove that there is only one continuous linear functional on this space, namely the trivial one (we write $(L^p)^* = \{0\}$).

A7. Here are some remarks on reflexive spaces. Let E be a normed space.

(A) Let $J : E \to E^{**}$ be defined with $(Jx)(\varphi) = \varphi(x)$. Prove that J is well-defined, incjective and isometry $\|Jx\| = \|x\|$.

(B) If J is surjective, we say that E is reflexive. Prove that any Hilbert space is reflexive.

(C) Suppose that E is reflexive. Prove that E is a Banach space.

A8. Prove that the map $T : l^1 \to (c_0)^*$ given with

$$(Ty)(x) = \sum_{i=1}^{\infty} x_i y_i$$

is well-defined, injective, surjective and isometry (i.e. $\|y\|_{l_1} = \|Ty\|_{(c_0)^*}$). Conclude that $(c_0)^* = l_1$.

Hahn-Banach Theorem (analytic)

B1. **(useful version)** Let $(X, \|\cdot\|)$ be a normed space and $M \subset X$ be a linear subspace. Let $g \in M^*$. Prove that there is a bounded linear functional $f \in X^*$ such that $g(x) = f(x)$ for $x \in M$ and $\|f\|_{X^*} = \|g\|_{M^*}$.

B2. **(duality formula)** Let $(X, \|\cdot\|)$ be a normed space. Prove that

$$\|x\| = \sup_{f \in X^* : \|f\| \le 1} f(x)$$

and the supremum above is attained.
Hint: First prove that for all $x_0 \in X$, there is $\varphi_{x_0} \in X^*$ such that

$$\varphi_{x_0}(x_0) = \|x_0\|^2 \text{ and } \|\varphi_{x_0}\| = \|x_0\|.$$

B3. Let $(X, \|\cdot\|)$ be a normed space. Prove that if $\varphi(x_1) = \varphi(x_2)$ for all $\varphi \in X^*$ then $x_1 = x_2$.

B4. Let $(E, \|\cdot\|)$ be a Banach space and $A \subset E$ be its subset. Suppose that for every $f \in E^*$, the set

$$f(A) = \{f(x) : x \in A\}$$

is bounded in $\mathbb{R}$. Prove that A is a bounded set in E (i.e. one can find a ball $B(0,R)$ for some $R > 0$ such that $A \subset B(0,R)$).

B5. Consider $L^p(\Omega, \mathcal{F}, \mu)$ with $1 \le p < \infty$ and $1/p + 1/q = 1$. Prove that

$$\|f\|_p = \sup_{g \in L^q : \|g\|_q \le 1} \int_X f(x) g(x) d\mu(x),$$

B6. Prove that the map $\Phi : L^1(0,1) \to (L^\infty(0,1))^*$ given with $(\Phi(f))(g) = \int_0^1 f(x)\, g(x)\, dx$ is well-defined (i.e. $\Phi(f) \in (L^\infty(0,1))^*$ for all $f \in L^1(0,1)$) but Φ is not surjective.
Remark: Roughly speaking, we say that $L^1(0,1) \subset (L^\infty(0,1))^*$ but $L^1(0,1) \neq (L^\infty(0,1))^*$.

B7. Prove that the map $\Phi : l^1 \to (l^\infty)^*$ given with $(\Phi(x))(y) = \sum_{i=1}^\infty x_i\, y_i$ is well-defined (i.e. $\Phi(x) \in (l^\infty)^*$ for all $x \in l^1$) but Φ is not surjective.
Remark: Roughly speaking, we say that $l^1 \subset (l^\infty)^*$ but $l^1 \neq (l^\infty)^*$.

B8. **(Banach limit)** Prove that there is a bounded functional on l^∞ denoted with φ such that

- $\varphi((x_0, x_1, x_2, ...)) = \varphi((x_1, x_2, x_3, ...))$, i.e. φ does not depend on finitely many terms,
- for $x \in l^\infty$ we have $\liminf_{n\to\infty} x_n \leq \varphi(x) \leq \limsup_{n\to\infty} x_n$,
- for converging $x \in l^\infty$ we have $\varphi(x) = \lim_{n\to\infty} x_n$.

Hint: consider subspace $W = \{x \in l^\infty : \lim_{n\to\infty} \frac{x_1+x_2+...+x_n}{n} \text{ exists}\}$. Observe that when $x_n \to \alpha$, we also have $\frac{x_1+x_2+...+x_n}{n} \to \alpha$.

Hahn-Banach Theorem (geometric)

C1. Let E be a normed space and $F \subset E$ be a linear subspace such that $\overline{F} \neq E$. Prove that there is $\varphi \in E^*$ such that $\varphi \neq 0$, $\|\varphi\| = 1$ and $\varphi(x) = 0$ for all $x \in F$.

C2. Let E be a normed space and $F \subset E$ be a linear subspace such that for all $\varphi \in E^*$

$$\forall_{x\in F}\ \varphi(x) = 0 \implies \varphi = 0.$$

Prove that F is dense in E.

C3. Let H be a Hilbert space and $M \subset H$ be its linear subspace. Prove that $(M^\perp)^\perp = \overline{M}$. In particular, when M is closed, $(M^\perp)^\perp = M$.

C4. Let X be a vector space (not necessarily normed or Banach) over $\mathbb{R}$. Let φ, φ_1, ..., φ_k be linear functionals on $\mathbb{R}$ (i.e. linear maps from X to $\mathbb{R}$). Suppose that

$$(\forall_{i=1,...,k}\ \varphi_i(v) = 0) \implies \varphi(v) = 0.$$

Prove that φ is a linear combination of $\varphi_1, ..., \varphi_k$, i.e. there are real numbers $\lambda_1, ..., \lambda_k$ such that $\varphi = \sum_{n=1}^k \lambda_n \varphi_n$. *Hint:* Study $F(x) = (\varphi_1(x), ..., \varphi_k(x), \varphi(x))$.

C5. (**Riesz Lemma**) Let $(X, \|\cdot\|)$ be a normed space and $M \subset X$ a closed (strictly contained) subspace. Prove that for any $\alpha \in (0,1)$ there is $x \in X$ such that $\|x\| = 1$ and $\text{dist}(x, M) \geq \alpha$.

C6. Prove that if X is finite dimensional, one can obtain Riesz Lemma for $\alpha = 1$. Prove that this is not possible, in general, for infinite dimensional X (study $X = l^\infty$).

C7. (**compactness of the ball**) Use Riesz Lemma to prove that if $(X, \|\cdot\|)$ is infinite dimensional space, ball $B_X = \{x \in X : \|x\| \leq 1\}$ is not compact.

C8. In the following exercise we will see that in infinite dimensional setting, *something* has to be assumed about two convex sets so that they can be separated (in finite dimensional case, convexity of both sets is sufficient). Let $E = l^1$ with its usual norm and consider two subsets:

$$X = \left\{x \in l^1 : x_{2n} = 0 \text{ for all } n \geq 1\right\}$$

$$Y = \left\{y \in l^1 : y_{2n} = \frac{1}{2^n} y_{2n-1} \text{ for all } n \geq 1\right\}.$$

(a) Check that X and Y are closed linear spaces in l^1. Verify that $\overline{X+Y} = E$.

(b) Consider sequence c defined with $c_{2n-1} = 0$ and $c_{2n} = \frac{1}{2^n}$. Check that $c \notin X + Y$.

(c) Set $Z = X - c$ and check that $Y \cap Z = \emptyset$. Can one separate Y and Z?

C9. Let $I : L^2(0,1) \to \mathbb{R}$ be a (nonlinear!) function defined with

$$I(u) = \int_0^1 |u(x)| \cos^2(x)\, \mathrm{d}x.$$

(A) Prove that I is continuous on $L^2(0,1)$.

(B) Prove that the set

$$\{(u,\lambda) \in L^2(0,1) \times \mathbb{R} : I(u) < \lambda\}$$

is open and convex.

(C) Fix $u \in L^2(0,1)$. Prove that there exists $v_u \in L^2(0,1)$ such that for all $w \in L^2(0,1)$ we have

$$I(u+w) \geq I(u) + \langle w, v_u \rangle.$$

What is v_u in the language of classical calculus for convex functions?