Functional Analysis (WS 20/21), Problem Set 9

(spectrum of operators on Hilbert spaces)[1]

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In what follows, let H be a **complex** Hilbert space.

Spectrum of T is the set $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not have a bounded inverse}\}$.
Resolvent of T is the set $\rho(T) = \mathbb{C} \setminus \sigma(T)$.

Resolvent is always an open subset of $\mathbb{C}$. Moreover, spectrum is always bounded with $|\sigma(T)| \leq \|T\|$ and so, it is compact in $\mathbb{C}$. Moreover, Liouville Theorem from analytic functions theory implies that the spectrum cannot be empty.

Helpful fact from the theory of Banach spaces: if $\|T\| < 1$ then operator $I - T$ is invertible. It's inverse is given by $(I - T)^{-1} = \sum_{k=0}^{\infty} T^k$.

1. Let A be a bounded operator. Prove that $\sigma(A)$ can be decomposed into three disjoint parts:

   (a) point spectrum: $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not injective,

   (b) continuous spectrum: $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is injective but not surjective and image of $A - \lambda I$ is dense in H,

   (c) residual spectrum: $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is injective but not surjective and image of $A - \lambda I$ is not dense in H

   If $\lambda \in \mathbb{C}$ belongs to the point spectrum, we say that λ is an eigenvalue of A.

2. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a complex matrix. Prove that A has a purely point spectrum.

3. Prove that $\sigma(A) \subset B(0, \|A\|) \subset \mathbb{C}$.

4. (**useful condition**) Let A be a bounded operator on Hilbert space H. Suppose there is a sequence $\{x_n\} \subset H$ and $\{\varepsilon_n\} \subset \mathbb{R}$ such that $\varepsilon_n \to 0$ and

$$\|Ax_n\| \leq \varepsilon_n \|x_n\|.$$

   Prove that A does not have a bounded inverse. In particular, it does not have an inverse as bounded linear isomorphisms have bounded inverses.

5. Let $M : L^2(0,1) \to L^2(0,1)$ be defined with $(Mf)(x) = xf(x)$. Find spectrum of M. Prove that it is purely continuous.

6. Let $A : l^2 \to l^2$ be defined with $Ax = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...\right)$. Find spectrum of A. Prove that it is purely residual.

7. Let p be a polynomial. Prove that if $\sigma(p(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}$.

[1] A useful reference for this topic is Chapter 9 of the book *Applied Analysis* by John Hunter and Bruno Nachtergaele available online at https://www.math.ucdavis.edu/ hunter/book/pdfbook.html. It may be helpful to read Wikipedia articles: "Spectrum (functional analysis)" and "Decomposition of spectrum (functional analysis)".

8. Let M be a multiplication operator from Problem 5. Find the spectrum of the operator
$$M^2 + M - 2.$$

9. Let $y \in l^\infty$ be a complex sequence and $T : l^2 \to l^2$ be defined with $Tx = (y_i\, x_i)_{i\in\mathbb{N}}$. Prove that $\sigma(T) = \overline{\{y_i : i \in \mathbb{N}\}}$.

10. Let $K \subset \mathbb{C}$ be a nonempty and compact subset. Construct a bounded linear operator $T : l^2 \to l^2$ such that $\sigma(T) = K$.

11. Let A be a bounded operator. We say that $\lambda \in \mathbb{C}$ is an *approximate eigenvalue* of A if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\|x_n\| = 1$ and $(A - \lambda I)x_n \to 0$. Prove that
    (a) if λ is an approximate eigenvalue of A, then $\lambda \in \sigma(A)$.
    (b) approximate spectrum contains point and continuous parts of spectrum.

12. Let G be a multiplication operator on $L^2(\mathbb{R})$ defined with $(Gf)(x) = g(x)f(x)$ for some bounded and continuous function g. Prove that
$$\sigma(G) = \overline{\{g(x) : x \in \mathbb{R}\}}$$
where upper line denotes the closure of the set. Can operator G have eigenvalues?

13. Consider the right shift operator R on $l^2(\mathbb{Z})$. Prove that $\sigma(R) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. What about left shift operator L?

14. Consider the right and left shifts operators on $l^2(\mathbb{N})$ (we usually denote this space with l^2) defined with
$$Rx = (0, x_1, x_2, ...), \qquad Lx = (x_2, x_3, x_4, ...).$$
Find $\sigma(R)$ and $\sigma(L)$.