Series 10, Riemann integrals

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Reminder:

In the previous exercises, we have been introduced to the notion of Newton's integral, which assigns to a function its primitive function. In particular, we can only calculate integrals of functions, which are the derivatives of other functions. This is a major constraint. For example, any derivative must have a Darboux property, hence even such a simple functions as piecewise constant ones, are somehow not integrable for us. Riemann's integrals are one of the possible generalizations of Newton's integrals. Intuitively, bounded, Riemann integrable functions are those, for which (introduced in previous exercises) Riemann sums always converge to the same limit. Hence, we try to define the integral of a function as a measure of the area under the function's curve. There are many characterizations of Riemann integrable functions. One of them states:

The bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is discontinuous on a set of measure zero.

Exercises:

(A1) Determine whether the function $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1, & \text{for } x \in A \cap [0, 1], \\ 0, & \text{otherwise,} \end{cases}$$

is Riemann integrable for $a)$ $A = \mathcal{C}$ (Cantor set), $b)$ $A = \mathbb{Q}$.

Hint: Cantor set is a compact subset of $[0, 1]$.

(A2) Show that for any $n \geq 1$

$$\frac{n^2\pi}{4} \leq \sum_{k=0}^{n} \sqrt{n^2 - k^2} \leq \frac{n^2\pi}{4} + n.$$

(A3) Let $f : [0, 1] \to \mathbb{R}$ be a C^1 function. Show that

$$\lim_{n\to+\infty} n \left(\frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) - \int_0^1 f(x)\,\mathrm{d}x \right) = \frac{f(1) - f(0)}{2}.$$

Use this fact to calculate the limit

$$\lim_{n\to+\infty} n \left(\frac{1^k + 2^k + ... + n^k}{n^{k+1}} - \frac{1}{k+1} \right).$$

Hint: Taylor's expansion of the integral

$$\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x)\,\mathrm{d}x.$$

(A4) Calculate the lateral surface area and the volume of

- the cylinder with height h and whose base has a radius r,
- the cone with height h and whose base has a radius r,

- the spheroid with $a = 3$ and $c = 5$ (this is a figure created by a revolution of an ellipse described by the equation $\frac{x^2}{c^2} + \frac{y^2}{a^2} = 1$).

(A5) Calculate the lateral surface area and the volume of a figure created by the revolution of a graph of the function $f(x) = \frac{1}{x}$ for $x \in [1, a]$. What happens when $a \to +\infty$?

(A6) Calculate the arc length of the curves parametrized with

- $\left[0, \frac{\pi}{4}\right] \ni t \mapsto (t, -\ln(\cos(t)))$,
- $[0, 1] \ni t \mapsto (t^3 + 2, 2t^3 + 3t^2 + t + 5)$.

(A7) Show that the area under the curve of $\cosh(x)$ on the interval $[a, b]$ is equal (in value) to the arc length of the curve of $\cosh(x)$ on the same interval.
Hyperbolic cosine is defined as $\cosh(x) = \frac{\exp(x)+\exp(-x)}{2}$.

(A8) Let $a \in \mathbb{R}$ and $b \in [0, 1]$. Show that

$$\sqrt{1 + a^2} - ab \geq \sqrt{1 - b^2}.$$

Let $f : [0, 1] \to \mathbb{R}$ be of C^1 class and the arc length of its curve on the interval $[0, 1]$ is equal to $f(1)$. Show that

$$\int_0^1 f(x)\,\mathrm{d}x \geq \frac{\pi}{4}.$$

For which function does the equality hold?