Series 12, Limits in multiple dimensions

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Exercises:

(A1) Calculate the limits

- $$\lim_{(x,y)\to(+\infty,+\infty)} \frac{x+y}{x^2-xy+y^2}$$,
- $$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)xy}$$,
- $$\lim_{(x,y)\to(0,0)} (x^2+y^2)^{xy}$$,
- $$\lim_{(x,y)\to(0,0)} \frac{\ln(\cos(x)-y^2)}{x^2+2y^2}$$,
- $$\lim_{(x,y)\to(0,0)} x\ln\left(x^2+2y^2\right)$$.

(A2) Determine whether the function $f:\mathbb{R}^2\to\mathbb{R}$

$$f(x,y)=\begin{cases} \frac{x^4-y^4}{x^4+y^4}, \text{ for } (x,y)\neq(0,0), \\ 0, \text{ otherwise,} \end{cases}$$

is continuous.

(A3) Find a function $g:\mathbb{R}\to\mathbb{R}$ for which a function $F:\mathbb{R}^2\to\mathbb{R}$ defined by

$$F(x,y)=\begin{cases} \frac{\sin(x^3-y^3)\exp(x+y)}{x-y}, \text{ for } x\neq y, \\ g(x), \text{ otherwise,} \end{cases}$$

is continuous.

(A4) Show that the double limit

$$\lim_{(x,y)\to(0,0)} (x+y)\sin\frac{1}{x}\sin\frac{1}{y}$$

exists, but the iterated limits

$$\lim_{x\to0}\lim_{y\to0}(x+y)\sin\frac{1}{x}\sin\frac{1}{y}, \quad \lim_{y\to0}\lim_{x\to0}(x+y)\sin\frac{1}{x}\sin\frac{1}{y}$$

do not.

(A5) Suppose the functions $f:\mathbb{R}^3\to\mathbb{R}$ is continuous and there exist points $v,w\in\mathbb{R}^3$ such that $f(v)>0>f(w)$. Show that the function f attains 0 infinitely many times.