Series 18

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26.05.2023

Reminder:
We say that a function $\mu^* : 2^X \to [0, +\infty]$ is an outer measure iff

- $\mu^*(\emptyset) = 0$,
- $\mu^*(A) \leq \mu^*(B)$ for any $A \subseteq B \subseteq X$,
- $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ for any $A_i \subseteq X$.

Let $\mathcal{F}$ be a given $\sigma-$field on X. We say that $\mu : \mathcal{F} \to [0, +\infty]$ is a measure on F iff

- $\mu(\emptyset) = 0$,
- $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ for any pairwise disjoint $A_i \in \mathcal{F}$.

Let $(X, \mathcal{F}, \mu)$ be a space with a given $\sigma-$field $\mathcal{F}$ and a measure μ defined on it. We say that $f : X \to \overline{\mathbb{R}}$ is measurable with respect to $\mathcal{F}$ iff

$$\{x \in X :, f(x) > a\} \in \mathcal{F} \text{ for any } a \in \mathbb{R}.$$

In particular if f, g are measurable functions, then the following sets are in $\mathcal{F}$.

- $\{x \in X :, f(x) < a\}$,
- $\{x \in X :, f(x) \leq a\}$,
- $\{x \in X :, f(x) \geq a\}$,
- $\{x \in X :, f(x) > g(x)\}$,
- $\{x \in X :, f(x) \geq g(x)\}$
- $\{x \in X :, f(x) = g(x)\}$.

And if f, g are measurable functions, then the following functions are measurable.

- $a\, f + b\, g$,
- $f\, g$,
- $|f|$,
- $\max\{f, g\}$
- $\min\{f, g\}$.

Exercises:

(A1) Find the point in the set

$$A = \{(x, y, z) \in \mathbb{R}^3 : \, 3x^2 + 3y^2 + 2xy + z^2 \leq 1\},$$

which is the furthest away from the origin $(0, 0, 0)$.

(A2) Let $f(x, y, z) = xy - z$. Find the supremum and the infimum of the given function on

a) $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$,

b) $A = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0,\ x^2 + y^2 + z^2 = 1\}$.

(A3) Let $n > 2$ be a natural number and $a, b > 0$ be such reals, that $a^2 < nb$. Suppose

$$\sum_{i=1}^{n} x_i = a, \quad \sum_{i=1}^{n} x_i^2 = b.$$

Find the biggest possible difference between the $\max\{x_i\}$ and $\min\{x_i\}$.

(A4) Let $X \subset \mathbb{R}$ be nonempty. Define for $A \subset \mathbb{R}$ a function

$$\mu_X(A) = \begin{cases} 1, & \text{if } A \subset X \\ 0, & \text{otherwise.} \end{cases}$$

Is μ_X an outer measure on $\mathbb{R}$?

(A5) Let $x_i \in X$ for $i = 1, 2, ...$ Define $\delta = \sum_{i=1}^{\infty} \delta_{x_i}$, where $\delta_{x_i}(A) = 1$ if $x_i \in A$ and $\delta_{x_i}(A) = 0$ otherwise. Determine, whether δ is an outer measure, measure or neither. If δ is a measure, what is the $\sigma-$field of $\delta-$measurable subsets of X?

(A6) Let $\mathcal{F}$ be a given $\sigma-$field on X and μ be a measure with respect to $\mathcal{F}$. For $A \subset X$ define $\mu^*(A) := \inf\{\mu(B) : B \in \mathcal{F}, A \subseteq B\}$. Prove that μ^* is an outer measure.

(A7) Let $f_x : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous/measurable functions. Show that

$$A := \{x \in \mathbb{R} : \{f_n(x)\}_{n \in \mathbb{N}} \text{ is increasing } \}$$

is a Borel/measurable set.

(A8) Let $f_1, f_2, f_3, f_4 : \mathbb{R}^n \to \mathbb{R}$ be measurable functions. Show that

$$f(x) = \begin{cases} f_1(x)\, f_2(x), & \text{if } f_3(x) > f_4(x), \\ |f_2(x)|, & \text{otherwise,} \end{cases}$$

is a measurable function.