Series 2, Taylor's expansion and Newton's method

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27.02.2023

Exercises:

(A1) Let $f : [a, b] \to \mathbb{R}$ be twice differentiable and $f'(a) = f'(b) = 0$. Show, that there exists $\xi \in (a, b)$ such that

$$|f''(\xi)| \geq \frac{4}{(b-a)^2} |f(b) - f(a)|.$$

(A2) Let $f : (0, +\infty) \to \mathbb{R}$ be twice differentiable and assume, that there exists x_0 such that $\sup_{x \in [x_0, +\infty)} |f''(x)| < +\infty$. Show that

$$\lim_{x \to +\infty} f(x) = 0 \Rightarrow \lim_{x \to +\infty} f'(x) = 0.$$

(A3) Let $f : (0, +\infty) \to \mathbb{R}$ be a C^2 function on its domain and

$$\lim_{x \to +\infty} x\, f(x) = 0, \qquad \lim_{x \to +\infty} x\, f''(x) = 0.$$

Show that

$$\lim_{x \to +\infty} x\, f'(x) = 0$$

(A4) Calculate the limit of the sequence

$$x_1 = \frac{1}{2}, \quad x_{n+1} = x_n(1 - \ln(x_n)).$$

(A5) Calculate the limit of the sequence

$$x_1 = -2, \quad x_{n+1} = x_n + \frac{2^{x_n+1} - 1}{\ln(2)}.$$

Moreover, find n such that x_n approximates its limit with an error of at most 2^{-15}.