Series 3, Taylor's expansion - repetition

Jakub Woźnicki

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Exercises:

(A1) Calculate the limits:

$$1) \lim_{x\to 0^+} \left(\frac{\sin(x)}{x}\right)^{\frac{1}{x^2}},$$
$$2) \lim_{x\to +\infty} x\left(\left(1+\frac{1}{x}\right)^x - e\right),$$
$$3) \lim_{x\to +\infty} \left(\frac{a^x-1}{x(a-1)}\right)^{\frac{1}{x}}, \quad a>0,\, a\neq 1$$

(A2) Determine whether the function

$$f(x) = \begin{cases} \frac{1}{x\ln(2)} - \frac{1}{2^x-1}, & \text{for } x\neq 0 \\ \frac{1}{2}, & \text{for } x = 0 \end{cases}$$

is continuous and differentiable in 0.

(A3) Let $f : [0,1] \to \mathbb{R}$ be twice differentiable such that $f(0) = f(1) = 0$ and $|f''(x)| \leq A$ for all $x \in (0,1)$. Show, that

$$|f'(x)| \leq \frac{A}{2}, \text{ for } x \in [0,1].$$

(A4) Let $f : [-1,1] \to \mathbb{R}$ be twice differentiable and $f(1) = 1 = 2f(-1)$, $f(0) = 0 = f'(0)$. Show, that there exists a point $\xi \in (-1,1)$ such that

$$f''(\xi) = \frac{5}{3}.$$