Series 4, Power series

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Reminder:

A series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z,\, z_0,\, a_n \in \mathbb{C}$$

is called a **power series** at z_0 with **coefficients** a_n. By **Cauchy–Hadamard formula** it is convergent in $\{z \in \mathbb{C} : |z - z_0| < R\}$, where

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

and divergent in $\{z \in \mathbb{C} : |z - z_0| > R\}$. Moreover it is absolutely and uniformly convergent in $\{z \in \mathbb{C} : |z - z_0| \leq r\}$ for $r < R$. If we define

$$S(z) := \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

then:

- S is well-defined in $\{z \in \mathbb{C} : |z - z_0| < R\}$,
- S is smooth in $\{z \in \mathbb{C} : |z - z_0| < R\}$ and

$$S^{(k)}(z) = \sum_{n=k}^{\infty} n \cdot (n-1) \cdot \ldots \cdot (n-k+1) a_n (z - z_0)^{n-k},$$

- $S^{(k)}(z_0) = k!\, a_k$.

Let us recall the following, known power series and their exact forms:

- $$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad -1 < x \leq 1$$
- $$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$$
- $$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}$$
- $$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad x \in \mathbb{R}$$
- $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

- $$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad |x| < 1.$$

Exercises:

(A1) Determine the set on which the following series converge:

$$1) \sum_{n=1}^{\infty} n^3 \, x^n, \qquad 2) \sum_{n=1}^{\infty} \frac{2^n}{n!} \, x^n,$$
$$3) \sum_{n=1}^{\infty} (2+(-1)^n)^n \, x^n, \qquad 4) \sum_{n=1}^{\infty} 2^n \, x^{n^2},$$
$$5) \sum_{n=1}^{\infty} 2^{n^2} \, x^{n!}.$$

(A2) Determine the set on which the following series converge:

$$1) \sum_{n=1}^{\infty} \frac{1}{2^n \, n^3} (x-1)^{2n}$$
$$2) \sum_{n=1}^{\infty} \frac{n}{n+1} \left(\frac{2x+1}{x} \right)^n.$$

(A3) Knowing that the number $R \in \mathbb{R}_+$ is a radius of convergence of the power series $\sum_{n=0}^{\infty} a_n \, x^n$ calculate the radius of convergence of

$$1) \sum_{n=0}^{\infty} 2^n \, a_n \, x^n,$$
$$2) \sum_{n=0}^{\infty} n^n \, a_n \, x^n.$$

(A4) Find the exact form of the following series:

$$1) \sum_{n=0}^{\infty} \frac{1}{2n+1} \, x^{2n+1},$$
$$2) \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1},$$
$$3) \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!},$$
$$4) \sum_{n=1}^{\infty} n \, x^n,$$
$$5) \sum_{n=1}^{\infty} n^2 \, x^n.$$

(A5) Find the Taylor series at $x_0 = 0$ of the following functions:

$$1) \quad f(x) = \sin(x^3), \quad x \in \mathbb{R},$$
$$2) \quad f(x) = \sin^4(x) + \cos^4(x), \quad x \in \mathbb{R},$$
$$3) \quad f(x) = \ln(1+x+x^2), \quad x \in (-1,1).$$

(A6) Find the Taylor series at $x_0 = 1$ of the following functions and calculate their radius of convergence.

$$1) \quad f(x) = (x+1)\exp(x),$$
$$2) \quad f(x) = \frac{1}{x}.$$