Series 5, Power series - continuation

Jakub Woźnicki

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Exercises:

(A1) Determine whether the function

$$1) \quad f(x) = \begin{cases} \frac{\exp(x)-1}{x}, & \text{when } x \neq 0, \\ 1, & \text{otherwise }. \end{cases}$$

$$2) \quad g(x) = \begin{cases} \frac{1-\cos(x)}{x^2}, & \text{when } x \neq 0, \\ \frac{1}{2}, & \text{otherwise }. \end{cases}$$

is $C^\infty(\mathbb{R})$. If so, calculate $f^{(n)}(0)$ and $g^{(n)}(0)$ for $n \in \mathbb{N}$.

(A2) Calculate

$$1) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

$$2) \quad \sum_{n=1}^{\infty} \frac{n}{2^{n-1}},$$

$$3) \quad \sum_{n=0}^{\infty} \frac{3^n(n+1)}{n!}.$$

(A3) Find a radius of convergence of the power series

$$\sum_{n=1}^{\infty} 2^{5^n} x^{a_n},$$

where $a_1 = 1$, $a_{n+1} = 5\,a_n + (-3)^n$, for $n \geq 1$.

(A4) Find a Taylor series of

$$1) \quad f(x) = \frac{1}{x^2+4x+7}, \text{ at } x_0 = -2,$$

$$2) \quad g(x) = \frac{1}{2+3x^2}, \text{ at } x_0 = 0,$$

$$3) \quad h(x) = \frac{x}{(x-2)(x-3)}, \text{ at } x_0 = 0,$$

$$4) \quad l(x) = \operatorname{arctg}\left(\frac{3x+8}{4x-6}\right), \text{ at } x_0 = 0,$$

$$5) \quad w(x) = \frac{\exp(x)}{1-x}, \text{ at } x_0 = 0.$$

(A5) Let $\{a_n\}$ be a sequence of non-negative numbers and let $R = 1$ be a radius of convergence of $f(x) = \sum_{n=1}^{\infty} a_n\, x^n$. Show that the limit $\lim_{x\to 1^-} f(x)$ exists if and only if $\sum_{n=0}^{\infty} a_n$ is convergent.

(A6) Assume that the radius of convergence of the sequence $f(x) = \sum_{n=0}^{\infty} a_n\, x^n$ exists and is equal to 1. Show that if $\lim_{x\to 1^-}(1-x)f(x)$ exists and is not equal to 0, then $\{a_n\}$ is not convergent to 0.