Series 6, Basics of uniform convergence

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Reminder:

We say that the sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions

$$\forall_{n\in\mathbb{N}} \quad f_n : A \longrightarrow \mathbb{R}$$

is uniformly convergent to a function

$$g : A \longrightarrow \mathbb{R}$$

iff

$$\sup_{x\in A} |f_n(x) - g(x)| \to 0, \text{ as } n \to +\infty. \tag{1}$$

Let us introduce the notation

$$\|f\|_\infty := \sup_{x\in A} |f(x)|,$$

then (1) can be written in the form

$$f_n \to g \text{ uniformly } \iff \lim_{n\to\infty} \|f_n - g\|_\infty = 0.$$

One can check, the following easy to prove qualities

- $\|f\|_\infty = 0 \Leftrightarrow f = 0$,
- $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$, for any $\alpha \in \mathbb{R}$,
- $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

Exercises:

(A1) Suppose that $f_n : A \to \mathbb{R}$, $g_n : A \to \mathbb{R}$, and $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ converge uniformly to f and g respectively. Prove that

1) $f_n + g_n$ converges uniformly to $f + g$ on A.
2) if both f and g are bounded on A, then $f_n \cdot g_n$ converges uniformly to $f \cdot g$ on A.

(A2) Suppose f_n converges uniformly to f on (a, b). Show that

$$\lim_{n\to\infty} \|f_n\|_\infty = \|f\|_\infty.$$

(A3) Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a uniformly continuous derivative f'. Prove that

$$n\left(f\left(x + \frac{1}{n}\right) - f(x)\right) \to f'$$

uniformly on $\mathbb{R}$.

(A4) Determine whether the following sequences converge uniformly on $[0, 1]$

$$\begin{aligned}
&1) \quad f_n(x) = \frac{1}{1 + (nx - 1)^2},\\
&2) \quad f_n(x) = \frac{x^2}{x^2 + (nx - 1)^2},\\
&3) \quad f_n(x) = x^n(1 - x),\\
&4) \quad f_n(x) = n\, x^n(1 - x).
\end{aligned}$$

(A5) Investigate pointwise and uniform convergence of the following series

$$\begin{aligned}
&1) \quad \sum_{n=1}^{\infty} \frac{x}{n^2}, \ \text{ on } \mathbb{R},\\
&2) \quad \sum_{n=1}^{\infty} x^2 \exp(-nx), \ \text{ on } (0, +\infty),\\
&3) \quad \sum_{n=1}^{\infty} x^2(1 - x^2)^{n-1}, \ \text{ on } [-1, 1].
\end{aligned}$$

(A6) Investigate the continuity and differentiability of the following function on $\mathbb{R}$

$$f(x) = \sum_{n=1}^{\infty} \operatorname{arctg}\left(\frac{x}{n^2}\right).$$

If it is differentiable, calculate $f'(0)$.