Series 7, Introduction to integration

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Reminder:

An indefinite integral of a function f is a family of its primitive functions. We write

$$\int f(x)\,\mathrm{d}x = F(x) + C,$$

where $F' = f$ and C is an arbitrary constant. Due to our knowledge of derivatives of elementary functions, we can derive:

- $$\int x^a\,\mathrm{d}x = \frac{1}{a+1}x^{a+1} + C, \text{ for } a \neq -1,$$
- $$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C,$$
- $$\int \cos(x)\,\mathrm{d}x = \sin(x) + C,$$
- $$\int \sin(x)\,\mathrm{d}x = -\cos(x) + C,$$
- $$\int \frac{1}{1+x^2}\,\mathrm{d}x = \operatorname{arctg}(x) + C,$$
- $$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin(x) + C,$$
- $$\int \exp(x)\,\mathrm{d}x = \exp(x) + C,$$
- $$\int \frac{1}{\cos^2(x)}\,\mathrm{d}x = \tan(x) + C.$$

The following identities are easy consequences of the known formulas for derivatives.

Linearity:

$$\int \alpha\, f(x) + \beta\, g(x)\,\mathrm{d}x = \alpha \int f(x)\,\mathrm{d}x + \beta \int g(x)\,\mathrm{d}x$$

for any f, g continuous and $\alpha, \beta \in \mathbb{R}$.

Integration by parts:

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

for any f, g differentiable.

Integration by substitution:

$$\int f(g(x))\, g'(x)\, \mathrm{d}x = F(g(x)) + C,$$

where $F' = f$, f is continuous and $g \in C^1$.

One may make use of the sophisticated substitutions below.

- If the function under the integral is a rational function of $\sin(x)$ and $\cos(x)$, then substitute,

$$t = \tan\left(\frac{x}{2}\right).$$

  This is the so-called Weierstrass substitution.

- If the functions under the integral is a rational function of x and $\sqrt{a\,x^2 + b\,x + c}$, then one can rewrite $a\,x^2 + b\,x + c$ in the canonical form, use linear substitution and obtain an integral of a rational function of y and

  1) $\sqrt{1 - y^2}$ - use trigonometric substitution $y = \sin(x)$,
  2) $\sqrt{y^2 - 1}$ - use hyperbolic substitution $y = \cosh(x)$,
  3) $\sqrt{1 + y^2}$ - use hyperbolic substitution $y = \sinh(x)$.

A definite integral of a continuous function $f : [a, b] \to \mathbb{R}$ is defined as

$$\int_a^b f(x)\, \mathrm{d}x = F(b) - F(a),$$

where $F' = f$.

Exercises:

(A1) Calculate the following integrals (integration by parts):

$$1)\quad \int x \operatorname{arctg}(x)\, \mathrm{d}x,$$
$$2)\quad \int \sin^4(x)\, \mathrm{d}x,$$
$$3)\quad \int \exp(3x)\, \sin(4x)\, \mathrm{d}x,$$
$$4)\quad \int x\, \exp(x) \sin(x)\, \mathrm{d}x.$$

(A2) Calculate the following integrals (integration of rational functions):

$$1)\quad \int \frac{x^4 + x^3 + x^2 + 4x - 1}{x^3 - x}\, \mathrm{d}x,$$
$$2)\quad \int \frac{2x + 5}{(x^2 + 4x + 5)^2}\, \mathrm{d}x,$$
$$3)\quad \int \frac{8}{x^4 + 4}\, \mathrm{d}x.$$

(A3) Calculate the following integrals (integration by substitution):

$$1)\quad \int \frac{x^2}{\sqrt{1+x^3}}\,\mathrm{d}x,$$
$$2)\quad \int \frac{1}{\exp(3x)+1}\,\mathrm{d}x,$$
$$3)\quad \int \sqrt{\frac{x-1}{x+1}}\,\mathrm{d}x,$$
$$4)\quad \int \sin^5(x)\,\mathrm{d}x.$$

(A4) Calculate the following integrals (Weierstrass substitution):

$$1)\quad \int \frac{1}{\sin(x)}\,\mathrm{d}x,$$
$$2)\quad \int \frac{1}{5+3\cos(x)}\,\mathrm{d}x,$$
$$3)\quad \int \frac{1+\sin(x)\cos(x)}{(2+\cos^2(x))(1+\sin^2(x))}\,\mathrm{d}x.$$

(A5) Calculate the following integrals (trigonometric/hyperbolic substitution):

$$1)\quad \int \sqrt{3-2x-x^2}\,\mathrm{d}x,$$
$$2)\quad \int \frac{x^2}{\sqrt{x^2-4x+3}}\,\mathrm{d}x,$$
$$3)\quad \int \frac{\sqrt{x^2+4x+9}}{x^2}\,\mathrm{d}x.$$

(A6) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that

- f is even $\iff \int_{-x}^{x} f(t)\,\mathrm{d}t = 2\int_0^x f(t)\,\mathrm{d}t$ for any $x \in \mathbb{R}$.
- f is odd $\iff \int_{-x}^{x} f(t)\,\mathrm{d}t = 0$ for any $x \in \mathbb{R}$.
- f is periodic with a period $T > 0 \iff \int_x^{x+T} f(t)\,\mathrm{d}t = \int_0^T f(t)\,\mathrm{d}t$ for any $x \in \mathbb{R}$.

(A7) Let $f \in C^2([0,1])$. Show that there exists $\xi \in (0,1)$ such that

$$\int_0^1 f(x)\,\mathrm{d}x = f(0) + \frac{1}{2}f'(0) + \frac{1}{6}f''(\xi).$$

Bonus exercise:

(Z1) Suppose $p, q \in \mathbb{N}$ are coprime. Show that

$$\int_0^1 \left(\{px\} - \frac{1}{2}\right)\left(\{qx\} - \frac{1}{2}\right) = \frac{1}{12pq},$$

where $\{x\}$ denotes the fractional part of a number x.