Series I, Taylor's expansion

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Reminder:

Let $f:(a,b)\longrightarrow\mathbb{R}$ be a $(n-1)-$times differentiable function in (a,b) and $n-$times at $x_0\in(a,b)$. Then, f admits a Taylor's expansion of order n at x_0. In particular, we may write f as

$$f(x)=f(x_0)+\sum_{k=1}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+o((x-x_0)^n)\text{ as }x\to x_0.$$

This is **Taylor's expansion with Peano's remainder**. If moreover, f is $(n+1)-$times differentiable in (a,b), then we may write

$$f(x)=f(x_0)+\sum_{k=1}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-x_0)^{n+1},\text{ where }\xi_x\text{ is between }x_0\text{ and }x.$$

This is **Taylor's expansion with Lagrange's remainder**. The notation $g(x)=o((x-x_0)^n)$ as $x\to x_0$ means that

$$\lim_{x\to x_0}\frac{g(x)}{(x-x_0)^n}=0.$$

Exercises:

(A1) Calculate $\exp(1/3)$ with an error of at most 10^{-4}.

(A2) Show that

$$1+\frac{1}{2}x-\frac{1}{8}x^2<\sqrt{1+x}<1+\frac{1}{2}x$$

for any $x>0$.

(A3) Determine whether the series

$$\sum_{n=1}^{\infty}\left(\ln(n+1)-\ln(n)-\frac{1}{n}\right)$$

is convergent.

(A4) Calculate the Maclaurin series of $f(x)=\ln(1+\sin(x))$ up to the fourth-order terms. Knowing it calculate further the Maclaurin series of

1) $g(x)=\ln(1-\sin(x)))$,
2) $h(x)=\ln(\cos^2(x))$.

(A5) Calculate the limits:

1) $\displaystyle\lim_{x\to0}\frac{x\operatorname{ctg}(x)-1}{x^2}$,

2) $\displaystyle\lim_{x\to0}\left(\frac{\operatorname{arctg}(x)}{x}\right)^{1/x^2}$,

3) $\displaystyle\lim_{x\to0}\frac{\arcsin(x)-x}{\operatorname{tg}(2x)-2\ln(1+x)-x^2}$.

(A6) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. Prove that

$$\lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = f''(x_0).$$

(A7) Show, that the opposite question to (A6) is untrue, i.e. that the existence of a limit in a previous exercise does not imply that f is twice differentiable at x_0.

(A8) Assume that $f : (a, +\infty) \to \mathbb{R}$ is twice differentiable. Let $M_i = \sup_{x \in (a,+\infty)} |f^{(i)}(x)|$, where $f^{(0)}(x) = f(x)$. Show that $M_1 \leq 2\sqrt{M_0 M_2}$.

(A9) Suppose that $f : [-1, 1] \to \mathbb{R}$ is thrice differentiable and

$$f(-1) = 0, \quad f(1) = 1, \quad f'(0) = 0.$$

Show, that $f^{(3)}(\xi) \geq 3$ for some $\xi \in (-1, 1)$.

(A10) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, such that $f(0) = 0, f'(0) = 1$, and $f''(0) < 0$. Define a sequence

$$x_1 = a \in \mathbb{R}, \quad x_{n+1} = f(x_n).$$

Show, that there exists a $\delta > 0$, such that if $0 < a < \delta$, then $\{x_n\}_{n \in \mathbb{N}}$ converges and its limit is equal to 0.

Bonus exercises:

(Z1) Assume $f : \mathbb{R} \to (0, +\infty)$ is twice differentiable. Determine whether there always exists a point $x_0 \in \mathbb{R}$ such that for any $x \in \mathbb{R}$

$$f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \geq 0.$$