

egzamin : pod koniec maja, gdzie lista tematów, albo rozłożony na dwie części marzec + maj.

Elliptic integrals:

sn - generalized sin (the function inverse to some integral)

$$x = \int_0^y \frac{dz}{\sqrt{1+mx^2+nx^4}}$$

$$\rightarrow n=0, m=-1 \Rightarrow$$

$$\int \frac{dz}{\sqrt{1-z^2}} = \arcsin$$



$$\rightarrow m=1, n=0 \Rightarrow$$

$$\int \frac{dz}{\sqrt{1+z^2}} \rightsquigarrow \operatorname{sh}$$



$$\rightarrow m=0, n=-1 \Rightarrow$$

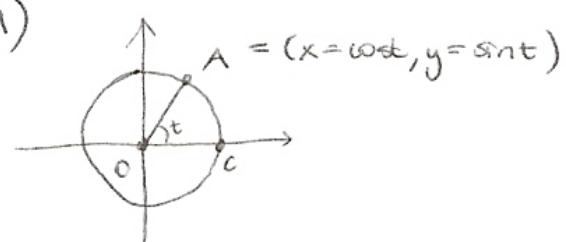
sl



lemniscate sn

$$\rightarrow m=-1-k^2, n=k^2, 0 < k < 1 \rightsquigarrow \operatorname{sn} \text{ Jacobian's sn}$$

1)



$$\cos^2 t + \sin^2 t = 1$$

$$\cos(-t) = \cos t$$

$$\sin(-t) = -\sin t$$

2π - periodic

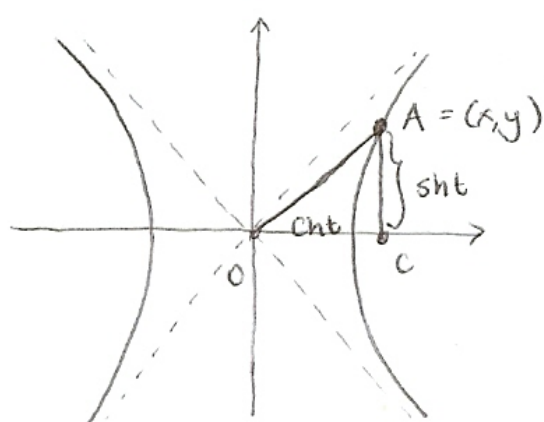
area of sector in the circle $S = \frac{1}{2} R^2 \alpha$



$$t = 2 \operatorname{area} (OCA)$$

$$t \in [0, 2\pi]$$

2)



$$x^2 - y^2 = 1$$

$$x = \operatorname{ch} t$$

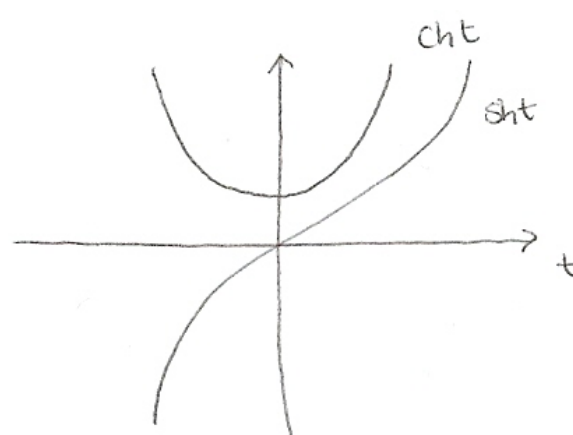
$$y = \operatorname{sh} t$$

$$\operatorname{ch}(-t) = \operatorname{ch} t$$

$$\operatorname{sh}(-t) = -\operatorname{sh} t$$

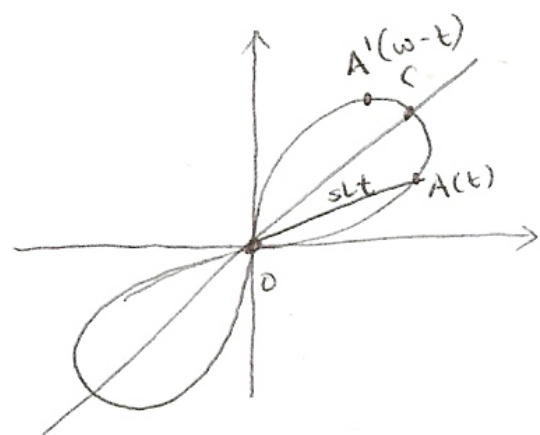
$$t = 2 \operatorname{area} (OCA)$$

$$t \in (-\infty, \infty)$$



$$3) (x^2 + y^2)^2 - 2xy = 0$$

(or in polar coordinates $x = r \cos \varphi, y = r \sin \varphi, r^2 = \sin 2\varphi$)



$$OA = r = \operatorname{sl} t \quad (\text{sinus lemniscatus})$$

$$\operatorname{sl}(-t) = -\operatorname{sl} t \quad - \text{odd}$$

$$\operatorname{cl} t := \operatorname{sl} \left(\frac{w}{2} - t \right), \text{ where } \frac{w}{2} - \text{length of arc } OC$$

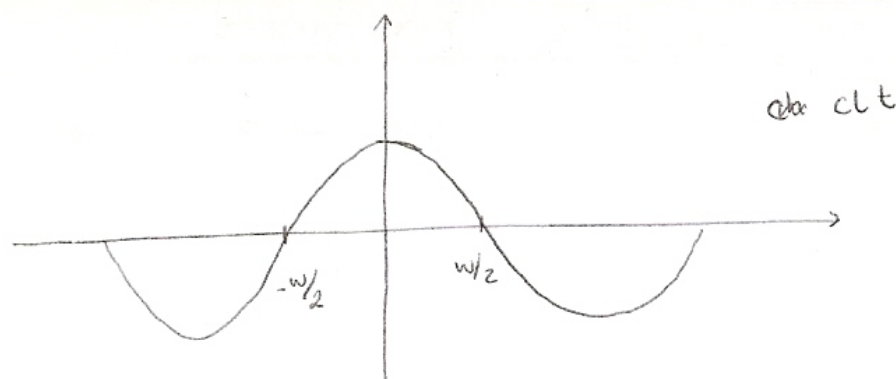
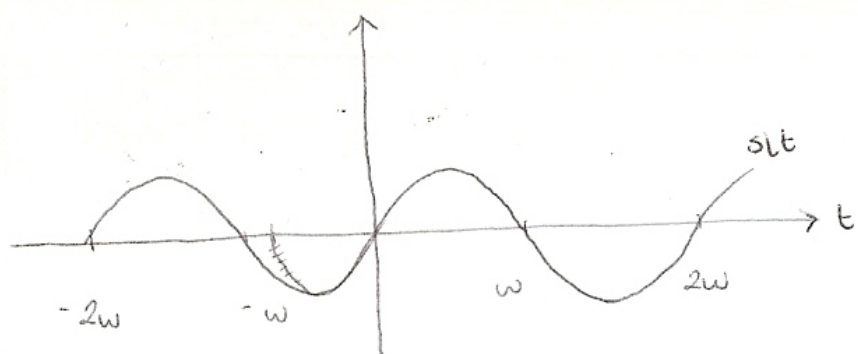
$$\operatorname{sl} \frac{w}{2} = 1 \quad (r=1)$$

length of lemniscat = $2w$

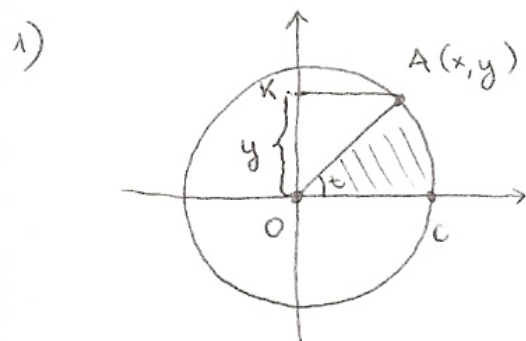
$$\operatorname{sl}(w-t) = \operatorname{sl} t$$

sl, cl - periodic with $2w$

cl - even function



The relation of these functions to the integrals:



$$t = 2 \text{area (OAC)}$$

$$y = \sin t$$

$$x = \sqrt{1-y^2}$$

$$\text{area (OAC)} = \text{area (OCAK)} - \text{area (}\triangle \text{OAK)}$$

$$\text{area (}\triangle \text{OAK)} = \frac{1}{2}yx = \frac{1}{2}y\sqrt{1-y^2}$$

$$\text{area OCAK} = \int_0^y \sqrt{1-y^2} dy = \text{integration by parts} = \frac{1}{2}y\sqrt{1-y^2} + \frac{1}{2} \int_0^y \frac{dy}{\sqrt{1-y^2}}$$

$$\text{area (OAC)} = \frac{1}{2} \int_0^y \frac{dy}{\sqrt{1-y^2}}$$

$$\Rightarrow t = \int_0^y \frac{dy}{\sqrt{1-y^2}}$$

$$y = \sin t$$

$$(\sin \frac{\pi}{2} = 1)$$

$$\int_0^1 \frac{dy}{\sqrt{1-y^2}} = \frac{\pi}{2}$$

if y increases from -1 to 1 $\Rightarrow$ t increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$\cos t := \sin(\frac{\pi}{2} - t)$$

$$t = \text{area (OCA)} = 2 (\text{area (OCAK)} - \text{area (}\triangle \text{OAK)})$$

$$t = \int_0^y \frac{dy}{\sqrt{1+y^2}} = \text{arcsinh } y$$

$$\text{if } \sqrt{1+y^2} + y = e^t$$

$$\Rightarrow y = \sinh t = \frac{e^t - e^{-t}}{2}$$

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

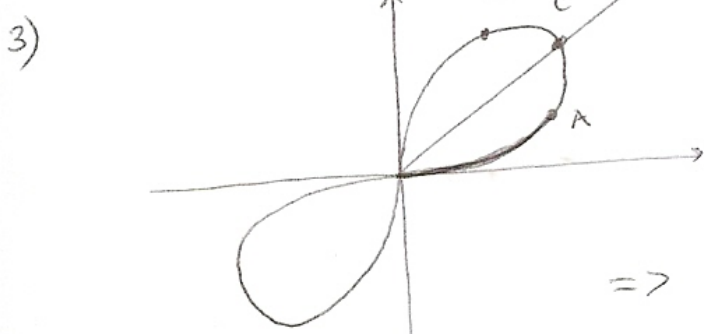
$$dt = \sqrt{(r d\varphi)^2 + dr^2} = \sqrt{r^2 \left(\frac{d\varphi}{dr}\right)^2 + 1} dr$$

$$r^2 = \sin 2\varphi \Rightarrow \varphi = \frac{1}{2} \arcsin(r^2)$$

$$\frac{d\varphi}{dr} = \frac{r}{\sqrt{1-r^4}}$$

$$\Rightarrow dt = \frac{dr}{\sqrt{1-r^4}}$$

$$\Rightarrow t = \int_0^y \frac{dr}{\sqrt{1-r^4}} \quad \frac{w}{2} = \int_0^1 \frac{dr}{\sqrt{1-r^4}}$$



Summary:

$y = \sin t$ defined as the function inverse to

$$\int_0^y \frac{dy}{\sqrt{1-y^2}}; -\frac{\pi}{2}, \frac{\pi}{2}$$

$$y = \sinh t$$

inverse to

$$\int_0^y \frac{dy}{\sqrt{1+y^2}}$$

$$t \in (-\infty, \infty)$$

$y = \sin t$ inverse to

$$\int_0^y \frac{dy}{\sqrt{1-y^4}}$$

$$t \in (-\frac{w}{2}, \frac{w}{2})$$

$$w = \int_0^1 \frac{dy}{\sqrt{1-y^4}}$$

$\Rightarrow$ so we are interested in properties of the function

$y = s(t)$ which is inverse to integral

$$t = \int_0^y \frac{dy}{\sqrt{1+my^2+ny^4}} \quad m, n \in \mathbb{R}$$

$s(t)$ is called generalized sin.

once again:

$$m = -1 \quad n = 0 \Rightarrow \sin$$

$$m = 1 \quad n = 0 \Rightarrow \sinh$$

$$m = 0 \quad n = -1 \Rightarrow \arctan$$

$$1+my^2+ny^4 = (1-k^2y^2)(1+k^2y^2) \Rightarrow \text{on } (t, k) \text{ Jacobi's elliptic function}$$

$$m = -(1+k^2), \quad n = k^2, \quad k \in (0, 1)$$

Properties: $P(y) := 1+my^2+ny^4$

$$t = \int_0^y \frac{1}{\sqrt{P(y)}} dy$$

• $P(y) = (1+qy^2)^2 \rightarrow \begin{matrix} q > 0 & \arctan \\ q < 0 & \log(\text{rat } f) \end{matrix}$
to avoid the cases $m^2 - 4n \neq 0$

• $P(y)$ has no real roots

either a) $1+mx+nz^2 \quad m^2 - 4n < 0$

or b) roots q $1+mx+nz^2$ are real and negative
 $m^2 - 4n > 0, \quad n > 0, \quad m > 0$

$$\int_0^y \frac{dy}{\sqrt{P(y)}} =: A \quad \text{integral - increasing function on real axis}$$

$$n = 0 \Rightarrow A = \infty$$

$$n \neq 0 \Rightarrow 0 < A < \infty$$

if $y \in (-\infty, \infty) \Rightarrow$ integral increases from $(-A)$ to A

$s(t)$ - defined on $(-A, A)$, increase from $(-\infty, \infty)$

Example: $\sinh, \quad A = \infty$

• $P(y)$ has 2 or 4 real roots

$$m^2 - 4n > 0 \quad \begin{matrix} n < 0 \\ n > 0, m < 0 \end{matrix}$$

Let us denote α - smallest positive root, in the interval $(-\alpha, \alpha)$ there are no other roots (because $-\alpha$ is also the root of $P(y)$)

$P(y) > 0$ in $(-\alpha, \alpha)$ (because $P(0) = 1$)

$\Rightarrow$ integral is defined and is an increasing function in the interval $(-\alpha, \alpha)$

$0 < a : = \int_0^a \frac{d\eta}{\sqrt{1+m\eta^2+n\eta^4}} \Rightarrow$ inverse function $y = s(t)$ is defined in the interval $(-a, a)$ increasing function from $-a$ to a .

examples:

sn : $\alpha = 1$
 $a = \pi/2$

sl : $a = 1$
 $a = \frac{w}{2}$

sn : $1 + m\eta^2 + n\eta^4 = (1-\eta^2)(1-k^2\eta^2)$

$\alpha = 1, a = \int_0^1 \frac{d\eta}{\sqrt{(1-\eta^2)(1-k^2\eta^2)}} =: K(k)$

complete elliptic function integral in the normal Legendre form

Remark: for sl

$$\frac{w}{2} = \int_0^1 \frac{d\eta}{\sqrt{1-\eta^4}} = \frac{1}{\sqrt{2}} K\left(\frac{1}{\sqrt{2}}\right)$$

$\Rightarrow$ sint : is defined on $(-\pi/2 \leq t \leq \pi/2)$

sl t : $(-w/2 \leq t \leq w/2)$

sn(t, k) $-K(k) \leq t \leq K(k)$

Ćwiczenia:

Elliptic integrals

$I = \int R(x, \sqrt{P(x)}) dx$ - sometimes can be elementary function
 $\left\{ \begin{array}{l} \text{polynomial of degree 3 or 4} \\ \text{rational function in both variables} \end{array} \right.$

Legendre's canonical form (elliptic integral of 1st ^{type} ~~order~~)

(1) $\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ modulus integral

2nd type

$$\int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}} dx$$

3rd type

$$\int \frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

After changing variables $x = \sin \varphi$ (1) $F(\varphi, k) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$ ^{mod}

complete elliptic integral of 1st type / kind: amplitude

$$F(\pi/2, k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = K(k)$$

in 1825 Legendre gave 1st tables of values of elliptic function integrals

$$k = \sin \alpha$$

$$\alpha, \varphi \in (0, 90^\circ)$$

and for "non-table" values one needs to use interpolation formulas.

• Calculations via series expansions

$$0 < k < 1$$

$$F(\varphi, k) = A_0 + \frac{1}{2} A_1 k^2 + \frac{1}{2} \cdot \frac{3}{4} A_2 k^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} A_3 k^6 + \dots$$

$$A_n = \int_0^\varphi \sin^{2n} \varphi d\varphi = \frac{2n-1}{2n} A_{n-1} - \frac{1}{2n} \cos \varphi \sin^{2n-1} \varphi, \quad n=0, 1, 2, \dots$$

$$F(\varphi, k) = A_0 \varphi - A_1 \sin 2\varphi + A_2 \sin 4\varphi - A_3 \sin 6\varphi + \dots$$

$$A_n = \frac{1}{n} \frac{2}{1+k'} \sum_{s=0}^{n-1} \frac{1 \cdot 3 \dots (2s-1)}{2 \cdot 4 \dots (2s)} \frac{1 \cdot 3 \dots (2s+2n-1)}{2 \cdot 4 \dots (2s+2n)} \frac{1}{k^{2s+n}}$$

$$k' = \sqrt{1-k^2}$$

$$\bar{k} = \frac{1-k'}{1+k'}$$

or recurrence relations:

$$A_n = \frac{(2n-2)^2}{n(2n-1)} \frac{k+k'^2}{k^2} A_{n-1} - \frac{(n-2)(2n-3)}{n(2n-1)} A_{n-2}$$

$$A_2 = \frac{2}{3} \frac{1+k'^2}{k^2} A_1 - A_0$$

these formulas are convenient for k small

There are other asymptotic formulas

Landen's transform:

$$\int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = \frac{1+k_1}{2} \int_0^{\varphi_1} \frac{d\varphi_1}{\sqrt{1-k_1^2 \sin^2 \varphi_1}}$$

$$k_1 = \frac{1-k'}{1+k'}, \quad k' = \sqrt{1-k^2} \quad \text{auxiliary modulus.}$$

$$\operatorname{tg}(\varphi_1 - \varphi) = k' \operatorname{tg} \varphi$$

$$\text{If } k = \sin \alpha$$

$$k_1 = \sin \alpha_1$$

Landen's transform is

$$\int_0^\varphi \frac{d\varphi}{\sqrt{1-\sin^2 \alpha \sin^2 \varphi}} = \frac{1}{2 \cos^2 \frac{\alpha}{2}} \int_0^{\varphi_1} \frac{d\varphi_1}{\sqrt{1-\sin^2 \alpha_1 \sin^2 \varphi_1}}$$

$$\sin \alpha_1 = \operatorname{tg}^2 \frac{\alpha}{2}, \quad \operatorname{tg}(\alpha_1 - \varphi) = \cos \alpha \operatorname{tg} \varphi$$

One can calculate elliptic integral of 1st kind by defining 2 sequences $\{\alpha_i\}$, $\{\varphi_i\}$ such that

$$\sin \alpha_{n+1} = \operatorname{tg}^2 \frac{\alpha_n}{2}$$

$$\operatorname{tg}(\varphi_{n+1} - \varphi_n) = \cos \alpha_n \operatorname{tg} \varphi_n$$

$$\alpha_n \rightarrow 0 \quad \text{for suff. leng} \quad \int_0^{\varphi_n} \frac{d\varphi_n}{\sqrt{1 - \sin^2 \alpha_n \sin^2 \varphi_n}} \approx \varphi_n$$

$$\int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - \sin^2 \alpha \sin^2 \varphi}} \approx \frac{\sqrt{\cos \alpha_1 \dots \cos \alpha_n}}{\sqrt{\cos \alpha}} \cdot \frac{\varphi_n}{2^n}$$

$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \approx (1 + k_1) \dots (1 + k_n) \frac{\pi}{2}$$

Gauss showed that

$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}} \stackrel{\text{l.t.}}{=} \int_0^{\pi/2} \frac{d\varphi_1}{\sqrt{a_1^2 \cos^2 \varphi_1 + b_1^2 \sin^2 \varphi_1}}$$

$$a_1 = \frac{a+b}{2}$$

$$b_1 = \sqrt{ab}$$

$\Rightarrow$ sequence of $\{a_n\}$, $\{b_n\}$

$$a_{n+1} = \frac{a_n + b_n}{2}$$

$$b_{n+1} = \sqrt{a_n b_n}$$

$$0 < k < 1$$

until $a_n \approx b_n$ for some n .

$$K(k) \approx \frac{\pi}{2a_n} \approx \left(\frac{\pi}{2b_n} \right)$$

If $k > 1$ in $\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}}$

$\Rightarrow$ change $x = \frac{y}{k}$

$$I = k_1 \int_0^y \frac{dy}{\sqrt{(1-y^2)(1-k_1^2 y^2)}} \quad k_1 = \frac{1}{k}$$

$$\int_1^b \frac{dx}{\underbrace{\sqrt{(1-x^2)(1-k^2 x^2)}}_{<0}} = \pm i \int_1^b \frac{dx}{\underbrace{\sqrt{(x^2-1)(1-k^2 x^2)}}_{\text{real}}}$$

$1 < b < 1/k$

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

2-valued function in $\mathbb{C}$ for $\varphi \in \mathbb{C}$

$1 - k^2 \sin^2 \varphi = 0$ - a lot of complex roots

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\operatorname{Re} \varphi = (2n + \frac{1}{2})\pi$$

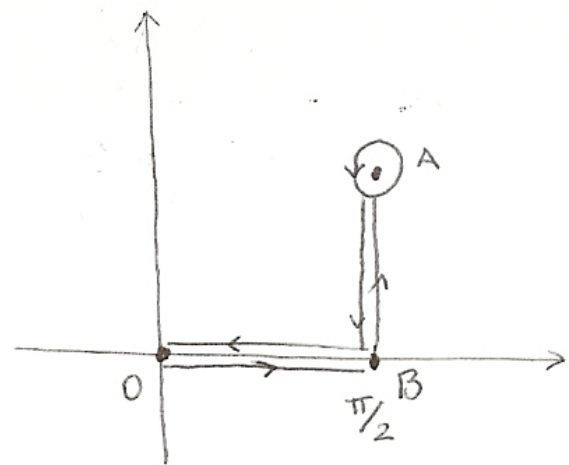
$$e^{ix} = A$$

winding around branch point

$$f(\varphi, k) = \frac{1}{\sqrt{1 - k^2 \sin^2 \varphi}} \rightarrow - \frac{1}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

$$\int_{\Gamma \subset \mathbb{C}} f(\varphi, k) d\varphi \quad \text{if } \Gamma \text{ closed and doesn't contain branch points} \Rightarrow \int_{\Gamma} = 0$$

if there is a branch point A inside Γ we deform curve



$$\Rightarrow \int_{\Gamma} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = 2K + 2iK'$$

$$K = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}$$

$$iK' = \int_{\pi/2}^{\pi/2 + i \operatorname{arcsinh} \frac{1}{k}} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} \quad \text{or} \quad K' = \int_0^{\pi/2} \frac{d\varphi'}{\sqrt{1-k'^2 \sin^2 \varphi'}}$$

$$k' = \sqrt{1-k^2}$$

$$\sin \varphi = \frac{1}{k} \sqrt{1-k'^2 \sin^2 \varphi'}$$

If only one initial point inside Γ

$$\int_{\Gamma} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = \pm 2K \pm 2iK'$$

$$\int_{\Gamma} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = \pm \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} + 2mK + 2inK'$$

single valued in the plane with cuts

"+" if $m, n \in 2\mathbb{Z}$

"-" if $m, n \in 2\mathbb{Z} + 1$

$$F(-\varphi, k) = -F(\varphi, k)$$

$$F(\varphi + \pi, k) = F(\varphi, k) + 2K$$

$$F(\pm \varphi \pm m\pi, k) = \pm F(\varphi, k) + 2mK$$

$m \in \mathbb{Z}$

$$x = \sin \varphi$$

$$\int_{\Gamma} \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} = \pm \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} + 2mK + 2niK'$$

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}}; \quad K' = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2 x^2)}}$$

for 2nd type elliptic integral

$$\int_0^x \sqrt{\frac{1-k^2 x^2}{1-x^2}} dx$$

$$x = \sin \varphi$$

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1-k^2 \sin^2 \varphi'} d\varphi'$$

- amplitude
modulus

$$E(-\varphi, k) = -E(\varphi, k)$$

$$E(\varphi + \pi, k) = E(\varphi, k) + 2E$$

where

$$E = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \varphi} d\varphi$$

$$|k| < 1$$

$$E(\varphi, k) = A_0 - \frac{1}{2} A_1 k^2 - \frac{1}{8} A_2 k^4 - \frac{1}{16} A_3 k^6 + \dots - \frac{1 \cdot 3 \cdot \dots \cdot (2n-3)}{2 \cdot 4 \cdot \dots \cdot (2n)} A_n k^{2n}$$

$$A_n = \frac{2n-1}{2n} A_{n-1} - \frac{1}{2\pi} \cos \varphi \sin^{2n-1} \varphi \int_0^{\pi} \frac{dx}{\sqrt{1-x^2}}$$

$$A_0 = \varphi$$

also F. exp

$$E(\varphi, k) = \sum_{n=0}^{\infty} B_n \sin n\varphi$$

Show that

$$\int_0^{\varphi} \frac{\sin^2 \varphi d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \frac{F(\varphi, k) - E(\varphi, k)}{k^2}$$

21.02.2011

Integration in the complex plane

Many properties can be derived if one considers $\mathbb{C}$

$$t = \tau + i\tau, \quad \tau, \tau \in \mathbb{R}$$

$$t = \int_0^w \frac{dz}{\sqrt{1 + mz^2 + nz^4}} = F(w), \quad w = u + iv \in \mathbb{C}, \quad m, n \in \mathbb{R}, \quad m^2 - 4n \neq 0, \quad n \neq 0.$$

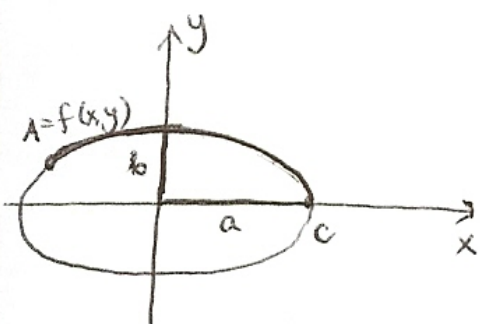
In principle we consider

$$\int_0^w R(z, \sqrt{a_0 + a_1 z + \dots + a_4 z^4}) dz$$

rational

Remark: why do we call those integrals elliptic?

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b \neq a$$



length of the arc CA

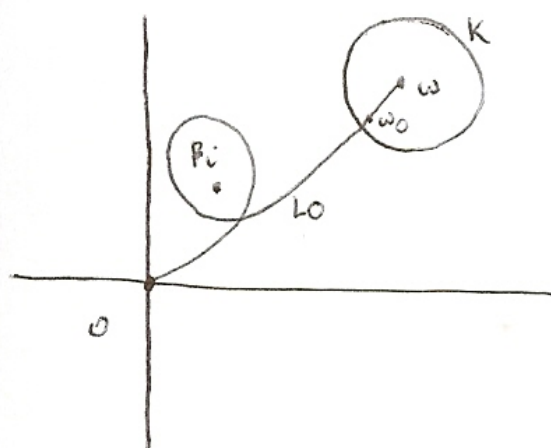
$$CA = \int_x^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_x^a \sqrt{1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}} dx =$$

$$= \frac{1}{a} \int_x^a \frac{a^4 - (a^2 - b^2)x^2}{\sqrt{(a^2 - x^2)(a^4 - (a^2 - b^2)x^2)}} dx$$

$$t = \int_0^w \frac{dz}{\sqrt{1 + mz^2 + nz^4}} = F(w)$$

the value of the integral on $\mathbb{C}$ depends on the path, because $F(w)$ -multivalued function.

$\beta_1, \beta_2, \beta_3, \beta_4$ - roots of $P(z) = 1 + mz^2 + nz^4$.



inside K there are no roots of $P(z)$.
Let us choose the path of integration inside K $F(w)$ will be ~~multivalued~~ ^{single} valued for γ_0

$$\frac{dt}{dw} = \frac{1}{\sqrt{1 + mw^2 + nw^4}}$$

$w = S(t)$ is single valued function of t .

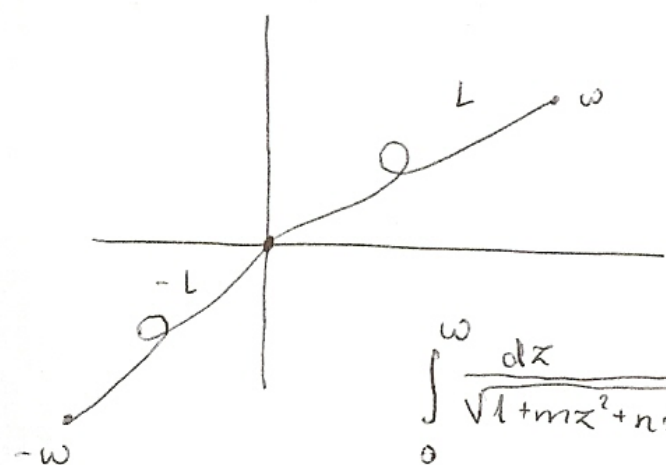
(this follows from Goursat's theorem: the values of the integral are different at different points)

but to a given w one gets infinitely many different t

$S(t)$ -periodic function of t .

let us prove a few properties

1) $\omega, -\omega$
 L, L' - symmetric with respect to $z=0$



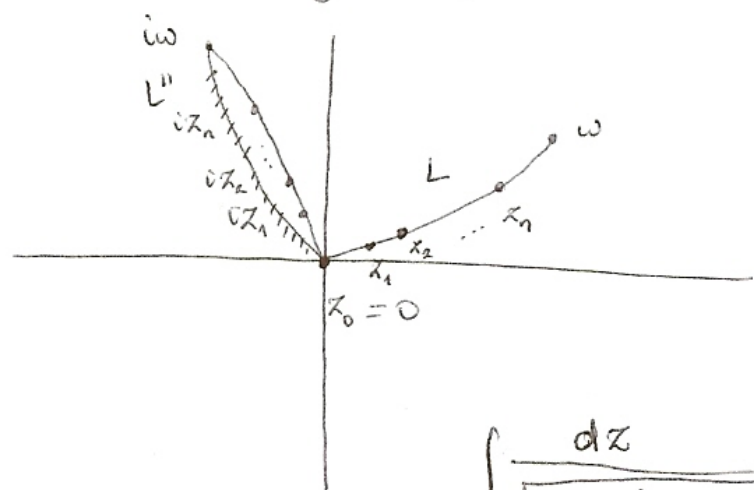
($-\omega, L'$ can be obtained from ω, L by rotating)

$$\int_0^{\omega} \frac{dz}{\sqrt{1+mz^2+nz^4}} = t \quad \text{inverse f.} \quad \leadsto \quad \omega = S(t)$$

$$\int_0^{-\omega} \frac{dz}{\sqrt{1+mz^2+nz^4}} = -t \quad S(-t) = -\omega$$

$$\Rightarrow S(-t) = -S(t), \quad S(t) \text{ is odd}$$

2) Rotation by angle $\pi/2$



$$\int_0^{\omega} \frac{dz}{\sqrt{P(z)}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{(z_k - z_{k-1})}{\sqrt{1+mz_k^2+nz_k^4}} =$$

$$= \int_L \frac{dz}{\sqrt{1+mz^2+nz^4}}$$

$$\int_{L''} \frac{dz}{\sqrt{1+mz^2+nz^4}} = \int_0^{i\omega} \frac{dz}{\sqrt{1+mz^2+nz^4}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{iz_k - iz_{k-1}}{\sqrt{1+m(iz_k)^2+n(iz_k)^4}} =$$

$$= i \int_0^{\omega} \frac{dz}{\sqrt{1-mz^2+nz^4}} = i \int_L \frac{dz}{\sqrt{1-mz^2+nz^4}}$$

$$\int_0^{i\omega} \frac{dz}{\sqrt{1+mz^2+nz^4}} = t \quad \text{inverse function} \quad i\omega = S(t)$$

$$\int_0^{\omega} \frac{dz}{\sqrt{1-mz^2+nz^4}} = t^* \quad \text{inv. funct.} \quad \omega = S^*(t^*)$$

$$t = it^*$$

$$\Rightarrow S(t) = S(it^*) = i\omega = iS^*(t^*)$$

$$\Rightarrow \boxed{S(it) = iS^*(t)}$$

$$\int_0^{\omega} \frac{dz}{\sqrt{1+mz^2+nz^4}} \quad S(t)$$

$$\int_0^{\omega} \frac{dz}{\sqrt{1-mz^2+nz^4}} \quad \omega = S^*(t)$$

$$\int_0^{\omega} \frac{dz}{\sqrt{1-z^2}} \quad \text{sn}(t)$$

$$\int_0^{\omega} \frac{dz}{\sqrt{1+z^2}} \quad \text{sh} t$$

$$\int_0^{\omega} \frac{dz}{\sqrt{1+z^2}} \quad \text{sht}$$

$$\int_0^{\omega} \frac{dz}{\sqrt{1-z^2}} \quad \text{sn}(t)$$

$$\int_0^{\omega} \frac{dz}{\sqrt{1-z^4}} \quad \text{sl} t$$

$$\int_0^{\omega} \frac{dz}{\sqrt{1-z^4}} \quad \text{sl} t$$

$$\int_0^{\omega} \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}} \quad \operatorname{sn}(t, k) \quad 0 < k < 1$$

$$\int_0^{\omega} \frac{dz}{\sqrt{(1+z^2)(1+k^2 z^2)}} \quad \operatorname{sn}^*(t, k)$$

$$\operatorname{sn}(it) = i \operatorname{sh}(t)$$

$$\operatorname{sh}(it) = i \operatorname{sn}(t)$$

$$\operatorname{sl}(it) = i \operatorname{ol}(t)$$

$$\operatorname{sn}(it, k) = i \operatorname{sn}^*(t, k)$$

Ex: $\operatorname{sn} t$ is odd function

$$\operatorname{sn}(-t) = \operatorname{sn}(i(it)) = i \operatorname{sh}(it) = i \cdot i \operatorname{sn} t = -\operatorname{sn} t$$

3) Change of variables

$$a) \quad t = \int_0^{\omega} \frac{dz}{\sqrt{1+mz^2+nz^4}} = \left\{ z = -\frac{\xi}{\zeta} \right\} = - \int_0^{\omega} \frac{d\xi}{\sqrt{1+m\xi^2+n\xi^4}} \quad \# -\omega = -\zeta(-t)$$

$$\Rightarrow S(-t) = -S(t)$$

$$b) \quad z = -i\xi$$

$$t = \int_0^{\omega} \frac{dz}{\sqrt{1-mz^2+nz^4}} = - \int_0^{\omega} \frac{id\xi}{\sqrt{1+m\xi^2+n\xi^4}}$$

$$\text{if } \omega = S^*(t), \quad i\omega = S(it)$$

$$\Rightarrow S(it) = i S^*(t)$$

c) if $\operatorname{sn}^*(t, k)$ is inverse function to

$$t = \int_0^{\omega} \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}} \quad 0 < k < 1$$

$$\left\{ z = \frac{\xi}{\sqrt{1-\xi^2}} \right\} \quad \int_0^{\omega} \frac{dz}{\sqrt{(1+z^2)(1+k^2 z^2)}} = \int_0^{\frac{\omega}{\sqrt{1+\omega^2}}} \frac{d\xi}{\sqrt{(1-\xi^2)(1-(1-k^2)\xi^2)}}, \quad k' = \sqrt{1-k^2} > 0$$

$$\frac{\omega}{\sqrt{1+\omega^2}} = \operatorname{sn}(t, k')$$

$$i \operatorname{sn}^*(t, k) = \operatorname{sn}(it, k)$$

$$\Rightarrow \operatorname{sn}(it, k) = i \frac{\operatorname{sn}(t, k')}{\sqrt{1-\operatorname{sn}^2(t, k')}}, \quad k' = \sqrt{1-k^2}$$

$$\sin(\alpha+\beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta = \sin\alpha \sqrt{1-\sin^2\beta} + \sqrt{1-\sin^2\alpha} \sin\beta$$

$$F(\sin(\alpha+\beta), \sin\alpha, \sin\beta) = 0$$

$$S(\alpha+\beta) = G(s(\alpha), s(\beta))$$

Weierstrass approach to elliptic functions:

e^z - periodic, $2\pi i$ - period
entire f.

$\sin z, \cos z$ - $2\pi i$ per.

$\operatorname{tg} z, \operatorname{ctg} z$ - meromorphic, $i\pi$ - periodic

ω - period $\neq 0$

$$e^{\frac{2\pi i}{\omega} z}$$

Rat. f. (per f) = periodic with the same period

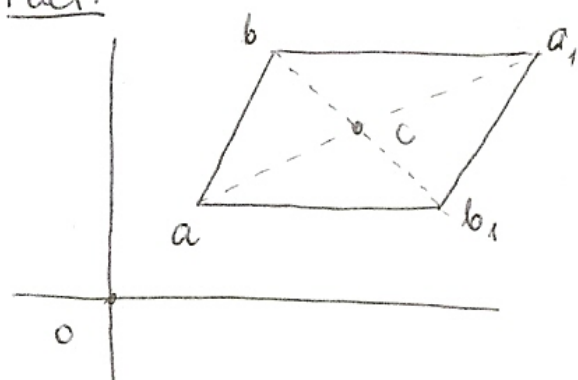
Def: $f(u)$ is elliptic if it is meromorphic and has two periods

$$f(u + \omega_1) = f(u)$$

$$f(u + \omega_2) = f(u)$$

$$\text{Im} \frac{\omega_2}{\omega_1} > 0$$

Fact:



$$c = \frac{a + a_1}{2} = \frac{b + b_1}{2}$$

assume that $f(u): \mathbb{C} \rightarrow \mathbb{C}$, $f(u + \omega) = f(u)$, $\omega \neq 0$

$\mathcal{R}(f)$ = set of all complex numbers, which are periods of $f(u)$

Proposition: $\Omega_{\mathcal{R}(f)}$ - group. if $\omega_i \in \mathcal{R}(f) \Rightarrow \omega_1 + \omega_2 \in \Omega_{\mathcal{R}(f)}$
 $i = 1, 2$ $-\omega_i \in \Omega_{\mathcal{R}(f)}$

Proof: $f(u + \omega_1 + \omega_2) = f(u + \omega_1) = f(u)$
 $f(u) = f(u - \omega_1 + \omega_2) = f(u - \omega_1) \in \Omega_{\mathcal{R}(f)}$

clearly $\omega_1, \dots, \omega_k \in \Omega_{\mathcal{R}(f)} \Rightarrow m_1 \omega_1 + \dots + m_k \omega_k \in \Omega_{\mathcal{R}(f)}$, $m_i \in \mathbb{Z}$

f - meromorphic $\neq$ constant.

Question: what might be the structure of $\Omega_{\mathcal{R}(f)}$? cases

1) $\Omega(f)$ has no accumulation points apart from ∞

2) Ω has

if accumulation point $\neq \infty \in \Omega(f) \Rightarrow \forall \varepsilon > 0 \exists \omega \in \Omega : 0 < |\omega| < \varepsilon$

$(\omega_1, \dots, \omega_k, \dots \rightarrow a \text{ as } k \rightarrow \infty) \Rightarrow \omega_{k+1} - \omega_k \in \Omega$
 $\{\omega_{k+1} - \omega_k\} \rightarrow 0$

Theorem: if $f(u)$ - meromorphic $\neq$ const

$\Rightarrow \Omega(f)$ has no accumulation points different from ∞ .
 (case 2 does not exist)

Legendre's result [1813]

$$\int \frac{dz}{\sqrt{P_4(z)}}$$

Riemann's surface



$$\mathbb{C} / \omega_1 + \tau \omega_2$$

$$Q = \int_0^{\sin \theta} \frac{ds}{\sqrt{1-s^2}}$$

elliptic integral 1st kind

$$w = S_n(u, k) \quad \text{inverse to} \quad u = \int_0^w \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$$

Legendre's theorem:

Any integral of $f(x) = R(x, \sqrt{P_4(x)})$ can be reduced to

$$\int \frac{dx}{\sqrt{P(x)}} \quad \text{or} \quad \int \frac{x^2 dx}{\sqrt{P(x)}} \quad \text{or} \quad \int \frac{dx}{(1+ax^2)\sqrt{P(x)}}$$

1st kind 2nd kind 3rd kind

Rat f = ratio of 2 polynomials = partial fractions decomposition

$$\text{If } f(x) = R(x, \sqrt{P_2(x)})$$

$$x = \frac{2u}{1+u^2} \Rightarrow f(x) = F(u) - \text{rational of } u.$$

Möbius transformation, frac.-linear

$$x \rightarrow \frac{ax+b}{cx+d}$$

degree 3 polynomial

$$\frac{1}{\sqrt{P_3(x)}} \xleftrightarrow{\text{ex.}} \frac{1}{\sqrt{P_4(x)}}$$

$$P(x) = P_4(x) = (1-x^2)(1-k^2x^2)$$

$$\text{if } r(x, w) \text{ is rational} = \underbrace{\frac{1}{2} [r(x, w) + r(x, -w)]}_{\text{even f. of } w} + \underbrace{\frac{1}{2} [r(x, w) - r(x, -w)]}_{\text{odd function of } w} =$$

$$= \underbrace{r_1(x, w^2)}_{\text{odd}} + \underbrace{r_2(x, w^2)}_{\text{even}} w$$

rational

$$\text{Let } w = \sqrt{P(x)}$$

$$\Rightarrow \int r(x, w) \text{ is reduced to } \int \text{rat f}(x) + \underbrace{\int r_3(x) \sqrt{P(x)}}_{\text{rat.}}$$

↓
we know how to solve it.

The same trick

Decompose $r_3(z)$ into odd and even parts:

$$\int \tilde{r}_1(x^2) \sqrt{P(x)} dx + \int \tilde{r}_2(x^2) \sqrt{P(x)} x dx = I_1 + I_2$$

$$r(x^2) \sqrt{\deg 2 \text{ pol}} \frac{dz^2}{2} - \text{this can be explicitly integrated.}$$

$I_1 =$ 1) integration by parts
2) recurrence relation

$$I_1 = \int \frac{R(x^2) dx}{\sqrt{P(x)}} = \left\{ \text{decompose } R(x^2) \right\}$$

$$\text{Terms: } J_n(x) = \frac{x^{2n}}{\sqrt{P(x)}}, \quad n=0, \pm 1, \quad K_n(x) = K_n(a, x) = \frac{1}{(1+ax^2)^n \sqrt{P(x)}}$$

Integration of J_{n+2} for $n \geq 0$ can be reduced to J_{n+1}, J_n :

$\int J_0 = 1st \text{ kind}$

$\int J_1 = 2nd \text{ kind}$

$$\begin{aligned} (z^{2n+1} \sqrt{P(z)})' &= (2n+1)z^{2n} \sqrt{P(z)} + \frac{1}{2} z^{2n+1} \frac{P'(z)}{\sqrt{P(z)}} = \frac{(2n+1)z^{2n} P(z) + \frac{1}{2} z^{2n+1} P'(z)}{\sqrt{P(z)}} \\ &= \frac{(2n+3)k^2 z^{2n+4} - (2n+2)(1+k^2)z^{2n+2} + (2n+1)z^{2n}}{\sqrt{P(z)}} \end{aligned}$$

From integration by parts

$$(2n+3)k^2 J_{n+2} = (2n+2)(1+k^2) J_{n+1} - (2n+1) J_n$$

$\Rightarrow$ every J_n is 1st or 2nd kind

Integration of $K_m(z)$ is reduced similarly to $K_n(z)$, $n < m$.

Addition theorem

$$t = \int_0^w \frac{dx}{\sqrt{1+mx^2+nx^4}} \quad w = s(t)$$

$$\alpha, \beta \in \mathbb{C}$$

$$u = s(\alpha)$$

$$v = s(\beta)$$

$$w = s(\alpha+\beta) = s(\gamma)$$

$$\alpha = \int_0^u \frac{dx}{\sqrt{1+mx^2+nx^4}}$$

Euler: Let $\alpha + \beta = \gamma = \text{const}$
vary

$$d(\alpha+\beta) = 0$$

$$\frac{du}{\sqrt{1+mu^2+nu^4}} + \frac{dv}{\sqrt{1+mv^2+nv^4}} = 0 \quad (1)$$

Equation (1) can be integrated and the first integral

is $F(u, v) = \text{const}$ - polynomial in two variables

$$F(u, 0) = u = s(\alpha) = s(\gamma) \quad v=0 \Rightarrow \beta=0$$

$$F(0, v) = v$$

$$C = F(u, v) \Big|_{\substack{\beta=0 \\ v=0}} = u \Big|_{\alpha=\gamma} = s(\gamma)$$

$$s(\alpha+\beta) = F(s(\alpha), s(\beta))$$

Consider

$$u^2 + v^2 + Au^2v^2 + 2Buv - C^2 = 0, \quad A, B, C - \text{constant}$$

$$(Av^2+1)u^2 + 2Buv + (v^2-C^2) = 0 \quad - \text{symmetric } u \leftrightarrow v$$

$$F(u, v)$$

Multiply by $Av^2 + 1$ and express u from equation above:

$$Auv^2 + Bv + u = \sqrt{C^2 + (B^2 + AC^2 - 1)v^2 - Av^4}$$

$$\Rightarrow v + Bu + Au^2v = \sqrt{C^2 + (B^2 + AC^2 - 1)u^2 - Au^4}$$

$F(u, v)$

$$dF(u, v) = \frac{\partial F}{\partial u}(u, v) du + \frac{\partial F}{\partial v}(u, v) dv =$$

$$= (2u + 2Auv^2 + 2Bv) du + (2v + 2Avu^2 + 2Bu) dv = 0$$

$$\Rightarrow \frac{du}{\sqrt{C^2 + (B^2 + AC^2 - 1)u^2 - Au^4}} = - \frac{dv}{\sqrt{C^2 + (B^2 + AC^2 - 1)v^2 - Av^4}}$$

Take

$$B^2 + AC^2 - 1 = mC^2$$

$$A = -nC^2$$

$$\Rightarrow \frac{du}{\sqrt{1 + mu^2 + nu^4}} + \frac{dv}{\sqrt{1 + mv^2 + nv^4}} = 0$$

→ has algebraic integral

After substitution:

$$u^2 - v^2 - nc^2 u^2 v^2 + 2\sqrt{1 + mc^2 + nc^4} uv - c^2 = 0$$

$$c = \frac{u\sqrt{1 + mv^2 + nv^4} + v\sqrt{1 + mu^2 + nu^4}}{1 - nu^2 v^2}$$

$$F(u, v)$$

$$F(u, 0) = u, \quad F(0, v) = v$$

$$s(\alpha + \beta) = f(s(\alpha), s(\beta))$$

$$\begin{aligned} s(\alpha + \beta) &= \frac{s^2(\alpha) - s^2(\beta)}{s(\alpha)\sqrt{1 + ms^2(\beta) + ns^4(\beta)} - s(\beta)\sqrt{1 + ms^2(\alpha) + ns^4(\alpha)}} \\ &= \frac{s(\alpha)c(\beta) + s(\beta)c(\alpha)}{1 - ns^2(\alpha)s^2(\beta)} \end{aligned}$$

$$\text{int: } m = -1, \quad n = 0$$

$$\sin(\alpha + \beta) = \sin \alpha \sqrt{1 - \sin^2 \beta} + \sin \beta \sqrt{1 - \sin^2 \alpha}$$

$$\text{sh t: } m = 1, \quad n = 0:$$

$$\sinh(\alpha + \beta) = \sinh(\alpha)\sqrt{1 + \sinh^2(\beta)} + \sinh(\beta)\sqrt{1 + \sinh^2(\alpha)}$$

$$\text{sl t: } m = 0, \quad n = -1$$

$$\text{sl}(\alpha + \beta) = \frac{\text{sl}(\alpha)\sqrt{1 - \text{sl}^4(\beta)} + \text{sl}(\beta)\sqrt{1 - \text{sl}^4(\alpha)}}{1 + \text{sl}^2 \alpha \text{sl}^2 \beta}$$

$$\alpha, \beta \in \mathbb{C}$$

$$\operatorname{sn} t: \quad m = -(1+k^2), \quad n = k^2, \quad 0 < k < 1$$

$$\operatorname{sn}(\alpha + \beta) = \frac{\operatorname{sn}^2 \alpha - \operatorname{sn}^2 \beta}{\operatorname{sn} \alpha \sqrt{1 + m \operatorname{sn}^2 \beta + n \operatorname{sn}^4 \beta} - \operatorname{sn} \beta \sqrt{1 + m \operatorname{sn}^2 \alpha + n \operatorname{sn}^4 \alpha}}$$

$$\sin(\sigma + i\tau) = \sin \sigma \operatorname{ch} \tau + i \operatorname{sn} \tau \cos \sigma \quad (\operatorname{cost} := \sqrt{1 - \operatorname{sn}^2 t})$$

$$\operatorname{sh}(\sigma + i\tau) = \operatorname{sh} \sigma \operatorname{cost} + i \sin \tau \operatorname{ch} \sigma$$

$$\operatorname{ch} t = \frac{e^t + e^{-t}}{2}, \quad \operatorname{sh} t = \frac{e^t - e^{-t}}{2}$$

$$e^t = \operatorname{ch} t + \operatorname{sh} t$$

$$e^{\alpha + \beta} = e^\alpha e^\beta \Leftrightarrow e^{\alpha + \beta} = \operatorname{ch}(\alpha + \beta) + \operatorname{sh}(\alpha + \beta) = \operatorname{odd} f = e^\alpha \cdot e^\beta$$

$$\operatorname{cn}(t, k) := \sqrt{1 - \operatorname{sn}^2(t, k)}$$

$$\operatorname{dn}(t, k) := \sqrt{1 - k^2 \operatorname{sn}^2(t, k)}$$

$$\operatorname{cn}(0, k) = \operatorname{dn}(0, k) = 1$$

$$\operatorname{sn}(0, k) = 0$$

$$\operatorname{sn}^2 + \operatorname{cn}^2 = 1$$

$$k^2 \operatorname{sn}^2 + \operatorname{dn}^2 = 1$$

$$\text{if } k=0 \Rightarrow \operatorname{sn} = \sin$$

$$\operatorname{cn} = \cos$$

$$\operatorname{dn} = 1$$

$$\operatorname{sn}(it, k) = i \frac{\operatorname{sn}(t, k')}{\sqrt{1 - \operatorname{sn}^2(t, k')}} \quad , \quad k' = \sqrt{1 - k^2}$$

2.03.2011

Theorem: If $f(u)$ - meromorphic and non constant then $\Omega(f)$ has no accumulation points (different from ∞)

Proof: Assume a - regular point of $f(u)$

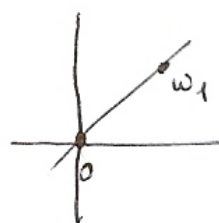
$\exists U_a$ st $\forall u \in U_a \quad f(u) \neq f(a) \Rightarrow$ none of the ω such that $0 < |u| < \varepsilon$ can be a period of $f(u)$.

if $\omega_k \in \Omega(f) \quad \omega_k \rightarrow \tilde{a} \in \mathbb{C}$

$\omega_{k+1} - \omega_k \rightarrow 0$ - accumulation point in $\Omega(f)$



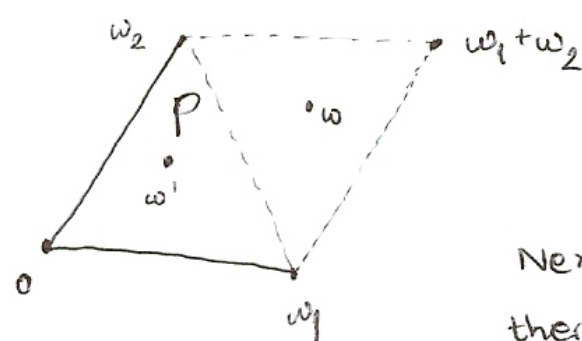
Case I: $\Omega(f)$ consist points which lie on line in $\mathbb{C}$



$\exists \omega_1$ which is the closest to origin (0,0)

$\omega_1 \in \Omega(f) \Rightarrow \omega = m\omega_1, m \in \mathbb{Z}$

Case II: Suppose that Ω contains at least 3 points (not on 1 line)



we chose the "closest" points to 0, that don't lie on one line: ω_1, ω_2

Next we show that in P and on ∂P there are no $\omega \in \Omega$

Assume that $\omega \in \Omega \cap P, \omega \in \Omega = \text{group} \Rightarrow \omega' = \omega_1 + \omega_2 - \omega \in P$, ω and ω' would be in different Δ s. Contradiction because by our choice ω_1 & ω_2 are "the closest" to 0.

Let ω be an arbitrary point in Ω

$$\omega = t_1 \omega_1 + t_2 \omega_2, \quad t_i \in \mathbb{R}$$

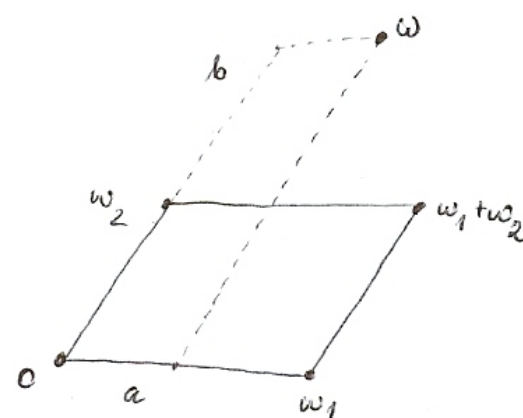
$$t_i = m_i + r_i, \quad m_i \in \mathbb{Z}, \quad r_i \in [0, 1)$$

$$\omega' = \omega - m_1 \omega_1 - m_2 \omega_2 = r_1 \omega_1 + r_2 \omega_2 \in P$$

$$r_1 = r_2 = 0$$

$$\omega = m_1 \omega_1 + m_2 \omega_2, \quad m_i \in \mathbb{Z}$$

$$\text{Im} \frac{\omega_2}{\omega_1} \neq 0$$



fundamental parallelogram

Corollary: All periods of a meromorphic periodic function can be expressed as $\omega = m_1 \omega_1 + m_2 \omega_2$ 2-periodic functions.

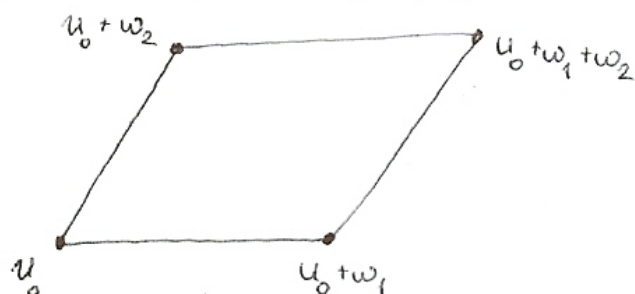
Fundamental parallelogram:

$$\omega_1, \omega_2, \quad \text{Im} \frac{\omega_2}{\omega_1} > 0$$

$$u \equiv v \pmod{(\omega_1, \omega_2)} \quad \text{means} \quad u = v + m_1 \omega_1 + m_2 \omega_2, \quad m_i \in \mathbb{Z}$$

$$u \equiv v, \quad u_1 \equiv v_1 \Rightarrow \forall n_1, n \in \mathbb{Z} \quad nu + n_1 u_1 \equiv nv + n_1 v_1$$

We can choose $u_0 \in \mathbb{C}$

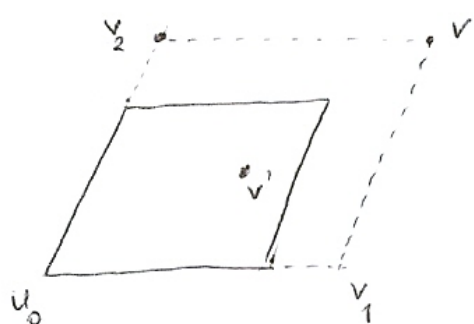


$$P(u_0) = \{ u \in \mathbb{C} : u = u_0 + r_1 \omega_1 + r_2 \omega_2, \quad 0 \leq r_i \leq 1 \}$$

$$P(o) := P$$

Theorem: $\forall v \in \mathbb{C}$ is congruent to one and only one point in $P(u_0)$ ($\forall u_0$).

Proof: $v = v_1 + v_2 - u_0 = (u_0 + t_1 \omega_1) + (u_0 + t_2 \omega_2) - u_0 = u_0 + t_1 \omega_1 + t_2 \omega_2$



$$t_i = m_i + r_i, \quad 0 \leq r_i \leq 1$$

$$v = u_0 + r_1 \omega_1 + r_2 \omega_2 + m_1 \omega_1 + m_2 \omega_2$$

$$v' = u_0 + r_1 \omega_1 + r_2 \omega_2$$

$$v \equiv v'$$



$$\mathbb{C} / m_1 \omega_1 + m_2 \omega_2 \quad m_i \in \mathbb{Z}$$

The field of elliptic functions

$$f(u) \text{ - elliptic} \quad \begin{cases} \text{meromorphic} \\ \omega_1, \omega_2 \text{ - period} \\ \text{Im} \frac{\omega_2}{\omega_1} > 0 \end{cases}$$

$K \stackrel{\text{def}}{=} \text{the set of all elliptic functions with given periods } \omega_1, \omega_2$

$K \neq \emptyset$ constant functions $\in K$.

We can check that K is a field

• if $\begin{matrix} f_1(u) \\ f_2(u) \end{matrix} \in K \Rightarrow f_1 \pm f_2 \in K$

• if $R(f_1(u), \dots, f_n(u)) \in K$ for any rational $R(x_1, \dots, x_n)$, $f_i(u) \in K$

• if $f(u) \in K \Rightarrow f'(u) \in K$

• if a - pole of $f(u) \in K$

$$f(u) = \sum_{k=-p}^{\infty} a_k (u-a)^k = \underbrace{g\left(\frac{1}{u-a}\right)}_{\text{principle part}} + h(u-a) \text{ - power series in } u-a$$

$$\forall a' \equiv a \pmod{(w_1, w_2)}$$

a' - pole of $f(u)$ with the same principle part

$$a' = a + \underbrace{m_1 w_1 + m_2 w_2}_{\tilde{w}}$$

$$\text{Let } u' = u + \tilde{w}$$

$$f(u') = f(u) = g\left(\frac{1}{u-a}\right) + h(u-a) = g\left(\frac{1}{u'-a'}\right) + h(u'-a')$$

$$u' - a' = u - a$$

f is meromorphic $\Rightarrow$ inside $P(u_0)$ there are only poles $a_1, \dots, a_r$
 (we write here a_j as many times as the order of the pole a_j)

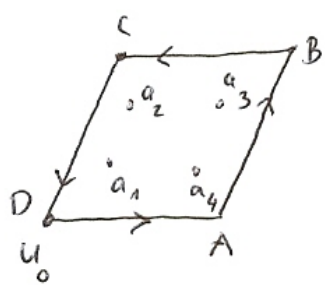
Def: r - order of the elliptic function

Clearly, all points congruent to $a_1, \dots, a_r$ are also poles of $f(u)$

General statements about elliptic functions:

Theorem: $f(u) \in K, r=0 \Rightarrow f \equiv \text{constant}$

Proof: in $P(u_0)$, $\max |f| < \infty$ any point in $\mathbb{C}$ is congruent to point in $P(u_0) \Rightarrow \max |f| < \infty$ in $\mathbb{C}$ + no poles ($r=0$)
 $\Rightarrow$ from Liouville's thm f is constant.



Choose u_0 such that on bound of $P(u_0)$

$\varphi(u)$ is continuous:

$$\int_{\partial P(u_0)} \varphi(u) du = \int_{u_0}^{u_0+w_2} (\varphi(u+w_1) - \varphi(u)) du - \int_{u_0}^{u_0+w_1} (\varphi(u+w_2) - \varphi(u)) du$$

on DA:

$$\int_{u_0}^{u_0+w_1} \varphi(u) du$$

on BC:

$$- \int_{u_0+w_2}^{u_0+w_1+w_2} \varphi(u) du = - \int_{u_0}^{u_0+w_1} \varphi(u+w_2) du$$

on AB:

$$\int_{u_0+w_1}^{u_0+w_1+w_2} \varphi(u) du = \int_{u_0}^{u_0+w_2} \varphi(u) du \dots$$

if $f(u)$ is elliptic $\Rightarrow \int_{\partial P(u_0)} f(u) du = 0 = 2\pi i$ (sum of the residues of $f(u)$ at points poles inside $P(u_0)$)

Theorem: The sum of residues of elliptic function $f(u) \in K$ for all poles inside $P(u_0)$ is equal to 0.

Theorem: $f(u) \in K$, $r=1 \Rightarrow$ only one pole of 1st order
 $\Rightarrow$ residuum = 0 $\Rightarrow r=0$

Theorem: $f(u) \in K$, order of f is r .

let c be fixed complex number

$a_1, \dots, a_r$ - poles $\Rightarrow f(u) = c$ for $u = u_1, \dots, u_r$ in $P(u_0)$
 $f(a_i) = \infty$

also $\frac{1}{2\pi i} \int_{\partial P(u_0)} \frac{f'(u)}{f(u)-c} du = \# \text{ zeros of } f - \# \text{ poles of } f$.

Theorem: Let $f(u) \in K$, $r \geq 2$

$b_1, \dots, b_r$ zeros of $f(u)$ in $P(u_0)$

$a_1, \dots, a_r$ poles of $f(u)$ in $P(u_0)$

$\Rightarrow b_1 + \dots + b_r = a_1 + \dots + a_r$

Proof: Consider $\varphi(u) = u \frac{f'(u)}{f(u)}$

$$\frac{1}{2\pi i} \int_{\partial P(u_0)} u \frac{f'(u)}{f(u)} du = \sum b_i - \sum a_i$$

$$(u+\omega_n) \frac{f'(u+\omega_n)}{f(u+\omega_n)} - \frac{u f'(u)}{f(u)} = \frac{\omega_n f'(u)}{f(u)}$$

$$\int_{\partial P(u_0)} \varphi(u) du = \int_{u_0}^{u_0+\omega_2} (\varphi(u+\omega_1) - \varphi(u)) du - \int_{u_0}^{u_0+\omega_1} (\varphi(u+\omega_2) - \varphi(u)) du = \int_{u_0}^{u_0+\omega_2} \frac{\omega_1 f'(u)}{f(u)} du - \int_{u_0}^{u_0+\omega_1} \frac{\omega_2 f'(u)}{f(u)} du =$$

$$\omega_1 \int_{u_0}^{u_0+\omega_2} \frac{f'(u)}{f(u)} du - \omega_2 \int_{u_0}^{u_0+\omega_1} \frac{f'(u)}{f(u)} du$$

$$\int_{u_0}^{u_0+\omega_1} \frac{f'(u)}{f(u)} du = \int_{u_0}^{u_0+\omega_1} d \log f(u) = \int_{\Sigma_0} d \log \xi = \log \xi \Big|_{\text{loop}} = 2\pi i k$$

$$\Rightarrow \frac{1}{2\pi i} \int_{\partial P(u_0)} u \frac{f'(u)}{f(u)} du = m_1 \omega_1 + m_2 \omega_2, \quad m_i \in \mathbb{Z}$$

$$\Rightarrow \sum b_i - \sum a_i = m_1 \omega_1 + m_2 \omega_2$$

$P(u)$ - Weierstrass function:

$r=2$ we need to construct elliptic function of order $r=2$

(eg) which has 1 pole of second order.

$$f(u) = \frac{A}{(u-a)^2} + \cancel{\frac{B}{(u-a)}} + \text{reg } f$$

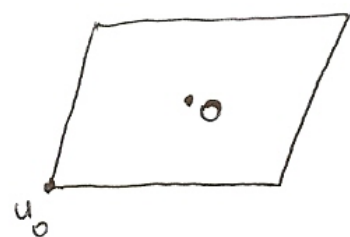
Let $u=0$ - pole of second order, with principle part $\frac{1}{u^2}$

$$f(u) \sim \frac{A}{u^2} + \text{reg part}$$

$$f(u) \sim \frac{1}{u^2} + \text{reg. part}$$

$$f(u) \approx \frac{1}{u^2} + \sum_0^\infty a_i u^i = \frac{1}{u^2} + h(u) \quad \text{we may assume}$$

$$f(u) \approx \frac{1}{u^2} + \sum_1^\infty a_i u^i = \frac{1}{u^2} + u \tilde{h}(u) = \frac{1}{u^2} + u \sum_0^\infty \tilde{a}_i u^i$$



$$p(u) := \frac{1}{u^2} + u \sum_0^\infty \tilde{a}_i u^i = \frac{1}{u^2} + u \tilde{h}(u)$$

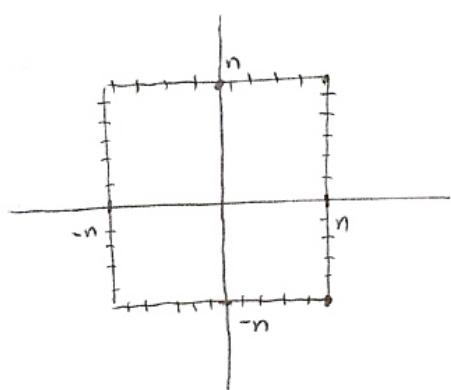
poles of $p(u)$ at $m_1 w_1 + m_2 w_2 = 0$

with principle part $g\left(\frac{1}{u-\tilde{\omega}}\right) = \frac{1}{(u-\tilde{\omega})^2}$

$$p(u) = \frac{1}{u^2} + \sum_{\substack{\tilde{\omega} = m_1 w_1 + m_2 w_2 \\ \tilde{\omega} \neq 0}} \left(\frac{1}{(u-\tilde{\omega})^2} - \frac{1}{\tilde{\omega}^2} \right)$$

Proposition:

$$S = \sum_{\substack{\tilde{\omega} \neq 0 \\ \tilde{\omega} = m_1 w_1 + m_2 w_2}} \frac{1}{|\tilde{\omega}|^3} < \infty$$



$$S_n = \sum_1^n \frac{1}{|w|^3} \leq 8n \cdot \frac{1}{(kn)^3} \approx \sum \frac{1}{n^2} < \infty$$

Proposition: $g(u) = \frac{1}{u} + \sum' \left(\frac{1}{u-\tilde{\omega}} + \frac{1}{\tilde{\omega}} + \frac{u}{\tilde{\omega}^2} \right) < \infty$

uniformly convergent infinite part

$g(u)$ - meromorphic with $u = \tilde{\omega}; 0$

This is uniformly convergent so we may differentiate

$$p(u) = -g'(u) = \frac{1}{u^2} + \sum' \left(\frac{1}{(u-\tilde{\omega})^2} - \frac{1}{\tilde{\omega}^2} \right)$$

Check that $p(u) = -g'(u)$ satisfies the following conditions:

1) poles at $0 \equiv \tilde{\omega} = m_1 w_1 + m_2 w_2$

with principle parts $\frac{1}{u^2}$ $\frac{1}{(u-\tilde{\omega})^2}$

2) $u \rightarrow 0$ in $\sum' \left(\frac{1}{(u-\tilde{\omega})^2} - \frac{1}{\tilde{\omega}^2} \right) \rightarrow$

3) $p(u)$ - periodic ?

$$p'(u) = -\frac{2}{u^3} - 2 \sum' \frac{1}{(u-\tilde{\omega})^3} = -2 \sum \frac{1}{(u-\tilde{\omega})^3}$$

$$p'(u + w_1) \Rightarrow \sum \frac{1}{(u+w_1-\tilde{\omega})^3} = \sum \frac{1}{(u+w_2-\tilde{\omega})^3} = \sum \frac{1}{(u-\tilde{\omega})^3}$$

$$\{\tilde{\omega} - w_1\} = \{\tilde{\omega} - w_2\}$$

$n=1, 2$

$$p'(u + w_n) = p'(u)$$

$$p(u + w_n) \neq p(u) = c_n$$

$p(u)$ - even

$$u = -\frac{1}{2}\omega_n$$

$$p\left(\frac{1}{2}\omega_n\right) - p\left(-\frac{1}{2}\omega_n\right) = 0 \Rightarrow c_n = 0$$

$\Rightarrow p(u)$ - periodic

$$p(u) = p(v)$$

$$u \equiv v$$

$$u \equiv -v$$

$$(p'(u))^2 = 4p^3 - g_2 p(u) - g_3 \quad \text{- next time}$$

3.03.2011

$$4) \quad p(u) = p(v) \Leftrightarrow u \equiv v, u \equiv -v$$

$p(u) - p(v)$ has a zero of 2nd order

if $u \equiv -v \Rightarrow 2v \equiv 0 \Rightarrow 2v$ is a period $\Rightarrow v \equiv 0$ or $v \equiv \frac{1}{2}\omega_1$ or

$$v \equiv \frac{1}{2}\omega_2 \text{ or } v \equiv \frac{1}{2}(\omega_1 + \omega_2)$$

$p(0) = \infty$ but 0 is the only pole $\mathbb{P} \Rightarrow$ all other values $u_0 \in \mathbb{P}$

$$e_1 = p\left(\frac{\omega_1}{2}\right), e_2 = p\left(\frac{\omega_1 + \omega_2}{2}\right), e_3 = p\left(\frac{\omega_2}{2}\right) \quad \text{pairwise diff}$$

$$p(u) = c < \infty \Rightarrow 2 \text{ solutions}$$

Differential equations $p(u)$:

$$p'(u) = -\frac{2}{u^3} + \underbrace{B(u)}_{\text{Taylor series at } u=0}$$

el. funct of order 3

($u=0$ is a pole of order 3)

$\Rightarrow p'(u)$ has 3 zeros in $\mathbb{P}$

$p(u) - p(v)$ has double zeros at $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1 + \omega_2}{2} \Rightarrow p'(u)$ has zeros at those points

$$p(u) \equiv p(v)$$

Let us define $f(u) = (p(u) - e_1)(p(u) - e_2)(p(u) - e_3)$

if $u=0 \Rightarrow f(u)$ has a 6-order pole at $u=0$

also at $u = \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1 + \omega_2}{2}$ $f(u)$ has double zeros

$$\Rightarrow Q(u) = \frac{(p'(u))^2}{f(u)} \quad \text{- elliptic, same periods as } p(u)$$

(from general theorem if in $\mathbb{P}$ no zeros, no poles $\Rightarrow$ const)

$$(p'(u))^2 \sim \frac{4}{u^6}, \quad f(u) \sim \frac{1}{u^6} \Rightarrow (p'(u))^2 = 4f(u)$$

Theorem:

$$p(u) \text{ satisfies } p'(u)^2 = 4(p(u) - e_1)(p(u) - e_2)(p(u) - e_3)$$

$$[(p'(u))^2 = 4p^3(u) - g_1 p(u) - g_2]$$

$$g(u) = \frac{1}{u} + \sum' \left(\frac{1}{u-\tilde{w}} + \frac{1}{\tilde{w}} + \frac{u}{\tilde{w}^2} \right) = \frac{1}{u} - \sum' \frac{u^2}{\tilde{w}^3} + \frac{u^3}{\tilde{w}^4} + \dots$$

$$g(u) = \frac{1}{u} - \sum_{n=1}^{\infty} \frac{e_n u^{2n}}{2n-1}$$

$$e_n = (2n-1) \sum' \frac{1}{\omega^{2n}}$$

$$0 = \sum' \frac{1}{\omega^n} \quad \text{if } n \text{ is odd}$$

$$p(u) = -\xi'(u)$$

$$p(u) = \frac{1}{u^2} + \sum_{n=2}^{\infty} c_n u^{2n-2}$$

$$p'(u) = -\frac{2}{u^3} + 2c_2 u + 4c_3 u^3 + \dots$$

$$(p'(u))^2 = \frac{4}{u^6} - \frac{8c_2}{u^2} - 16c_3 + \dots$$

$$p^3(u) = \frac{1}{u^6} + \frac{3c_2}{u^2} + 3c_3 + \dots$$

$$(p'(u))^2 - 4p^3(u) = -\frac{20c_2}{u^2} - 28c_3 + \dots$$

$$\text{here eliminated } \sim \frac{1}{u^6}$$

$$(p'(u))^2 - 4p^3(u) + 20c_2 p(u) = \text{holomorphic} \quad \text{eliminate terms} \\ \sim \frac{1}{u^2}$$

ell. functions no poles $\Rightarrow$ const

$$(p'(u))^2 - 4p^3(u) + 20c_2 p(u) = -28c_3$$

$$(p'(u))^2 - 4p^3(u) + \underbrace{20c_2 p(u)}_{g_2} + \underbrace{28c_3}_{g_3} = 0$$

$$g_2 = 20c_2 = 60 \sum' \frac{1}{\omega^4}, \quad g_3 = 28c_3 = 140 \sum' \frac{1}{\omega^6}$$

g_2, g_3 are called invariants

$$4x^3 - g_2 x - g_3 = 4(x-e_1)(x-e_2)(x-e_3)$$

$$\Delta = 16(e_1 - e_2)^2 (e_1 - e_3)^2 (e_2 - e_3)^2 \neq 0$$

$$\sum_{i=1}^3 e_i = 0$$

$$e_1 e_2 + e_2 e_3 + e_3 e_1 = -\frac{1}{4} g_2$$

$$e_1 e_2 e_3 = -\frac{1}{4} g_3$$

$$\left. \begin{array}{l} \text{solving} \\ \Rightarrow \Delta = g_2^3 - 27g_3^2 \end{array} \right\}$$

Let us find a recurrence relation for e_n (coeff. in expansion of $p(u)$)

$$2p'p'' = (12p^2 - g_2)p'$$

$$\Rightarrow p'' = 6p^2 - \frac{1}{2}g_2 = 6p^2 - 10c_2$$

$$p(u) = \frac{1}{u^2} + \sum_{n=2}^{\infty} c_n u^{2n-2}$$

$$p''(u) = \frac{6}{u^4} + \sum_{n=2}^{\infty} (2n-2)(2n-3) c_n u^{2n-4} - 10c_2 + 6p^2 u =$$

$$= -10c_2 + \frac{6}{u^4} + 12 \sum_{n=2}^{\infty} c_n u^{2n-2} + 6 \sum_{n=2}^{\infty} u^{2n-4} \sum_{k=2}^{n-2} c_k c_{n-k}$$

$$u^{2n-4}: ((2n-2)(2n-3) - 12)c_n = 6 \sum_{k=2}^{n-2} c_k c_{n-k} \quad n=4, 5, 6, \dots$$

$$(n-3)(2n+1)c_n = 3 / c_2 c_{n-2} + c_3 c_{n-3} + \dots + c_{n-2} c_2 \quad n=4, 5, 6, \dots$$

$\Rightarrow c_n$ is expressed in terms of c_2, c_3 for g_2, g_3

$$g_2 = \sum' \frac{1}{\tilde{w}^4}$$

$$c_4 = \frac{1}{3} c_2^5, \quad c_5 = \frac{3}{11} c_2 c_3, \dots$$

$$g_3 = \sum' \frac{1}{\tilde{w}^6}$$

$$c_6 = \frac{1}{13} \left(\frac{2}{3} c_2^3 + c_3^2 \right) \dots$$

$$G_{2n} = \sum' \frac{1}{\tilde{w}^{2n}} \quad \text{expressed in terms of } G_4, G_6$$

$$\tilde{w} = m_1 w_1 + m_2 w_2$$

by means of polynomials with rational positive coefficients

Summary:

- p satisfies a 1st order 2nd degree ODE
- this ODE can be derived by 2 methods
- expressed for series G_{2n}

Addition theorem for $p(u)$

we say that for $\varphi(u)$ there is an addition theorem

if $\forall u_1, u_2 \quad \varphi(u_1), \varphi(u_2), \varphi(u_1+u_2)$ satisfies some algebra equation

and this doesn't depend on u_1, u_2

$$e^{x_1+x_2} = e^{x_1} e^{x_2}$$

$$\varphi(x_1+x_2) = \varphi(x_1) \varphi(x_2)$$

Take $f(u) = p'(u) - ap(u) - b$ fix u_1, u_2

and choose a, b s.t. $f(u_1) = 0, f(u_2) = 0$

denote

$$p(u_i) = p_i, \quad p'(u_i) = p'_i$$

$$ap_1 + b = p'_1$$

$$ap_2 + b = p'_2$$

at $u=0$ $f(u)$ has 3 order pole

$\Rightarrow u_1, u_2$ - by our choice are zeros of $f(u)$ at $\mathbb{H}$

$$-(u_1+u_2) \bmod (w_1, w_2)$$

$$\sum \text{poles} = \sum \text{zeros}$$

denote

$$p(u_1 + u_2) = p_3$$

$$p'(u_1 + u_2) = p_3'$$

$p(u)$ - even

$p'(u)$ - odd

$$ap_3 + b = -p_3'$$

$$4x^3 - g_2x - g_3 = (p')^2$$

$$p'(u) = g_2 p(u) + b$$

$$4x^3 - g_2x - g_3 = (ax+b)^2$$

has roots p_1, p_2, p_3

$$p_i \neq p_j$$

$$p_1 + p_2 + p_3 = \frac{a^2}{4}$$

$$p_1 p_2 p_3 = \frac{b^2}{4} + \frac{g_3}{4}$$

$$p_1 p_2 + p_2 p_3 + p_3 p_1 = -\frac{ab}{2} - \frac{g_2}{4}$$

$$(1) \Rightarrow a = \frac{p_1' - p_2'}{p_1 - p_2}$$

$$b = \frac{p_1 p_2' - p_2 p_1'}{p_1 - p_2}$$

/ diff eq.

$$p(u_1 + u_2) = -p(u_1) - p(u_2) + \frac{1}{4} \left(\frac{p'(u_1) - p'(u_2)}{p(u_1) - p(u_2)} \right)^2$$

valid if $p(u_1), p(u_2), p(u_1 + u_2)$ are distinct but in fact
true for any u_1, u_2

16.03.2011

Expression of arbitrary elliptic function in terms of $p(u)$

$f(u)$ - arbitrary elliptic function, w_1, w_2

we need to show that $f(u) = F(p(u; w_1, w_2))$

1) $f(u)$ - even $f(u_1) = 0 \Leftrightarrow f(-u_1) = 0$
 $= \infty \quad \quad \quad = \infty$

if $u_1 \equiv -u_1 \Rightarrow$ zeros and poles are of even order

if $b_1, \dots, b_{2k}$ - zeros of $f(u)$ in $\mathbb{P}$
 $a_1, \dots, a_{2k}$ - poles including multiplicity

we take one representative of each pair $\pm b_s, \pm a_s$.

$\Rightarrow b_1, \dots, b_k$ - zeros
 $a_1, \dots, a_k$ - poles

$$Q(u) := \frac{(p(u) - p(b_1)) \dots (p(u) - p(b_k))}{(p(u) - p(a_1)) \dots (p(u) - p(a_k))}$$

has $\pm b_k$ zeros, $\pm a_k$ poles, p -Weierstrass function

- this is elliptic

$\frac{f(u)}{Q(u)}$ - elliptic, no poles, no zeros $\Rightarrow$ constant.

$\Rightarrow f(u) = \text{const} \cdot \underbrace{Q(u)}_{\text{expression in terms of } p(u)}$

Theorem: Every even elliptic function with period w_1, w_2 is a rational function of $p(u; w_1, w_2)$

2) $f(u)$ - arbitrary odd function

$p'(u)$ - odd $\Rightarrow \frac{f(u)}{p'(u)}$ - even $\Rightarrow$ case 1

Theorem: Any odd elliptic function is $p'(u) \cdot$ rational function of $p(u)$

3) f - any elliptic function

$$f = \underbrace{\frac{1}{2}(f(u) + f(-u))}_{\text{even}} + \underbrace{\frac{1}{2}(f(u) - f(-u))}_{\text{odd}}$$

Theorem: $f(u) = R(p(u)) + p'(u) \cdot R_1(p(u))$

The structure of elliptic function

Example $(p(u))^{(2n)}$

$$(p')^2 = 4p^3 - g_2 p - g_3 \Rightarrow 2p'p'' = (12p^2 - g_2)p' \Rightarrow 2p'' = 12p^2 - g_2$$

$$p''' = 12pp'; p''' = 12p'^2 + 12pp'' = P(p(u))$$

$\Rightarrow \frac{(p(u))^{(2n)}}{(p(u))^{(2n+1)}} = \frac{P_{2n}(p(u))}{P_{2n+1}(p(u)) \cdot p'(u)}$ - leading coefficient $(2n+1)!/2$

$$P_{n+1}(p) = P_n'(p) \left(6p^2 - \frac{g_2}{2} \right) + \tilde{P}_n'(p) (4p^3 - g_2 p - g_3)$$

Let $f(u)$ be elliptic function such that in $\mathbb{P}$ it has only one

pole at $u=0$

$$\Rightarrow f(u) = P_1(p(u)) + p'(u)P_2(p(u))$$

Proof: Consider $f(u) + f(-u) = \underbrace{2P_1(p(u))}_{\text{if } = \infty \text{ at some point } u \neq 0}$

$\Rightarrow P_1$ should be polynomial

$$f(u) - f(-u) = 2p'(u)P_2(p(u)) \Rightarrow \text{similar argument } P_2 - \text{polynomial}$$

Consider $p(nu)$, $n \in \mathbb{Z}$ it is an ^{even} elliptic function, with periods w_1, w_2

Addition theorem:

$$p(u_1 + u_2) = -p(u_1) - p(u_2) + \frac{1}{4} \left(\frac{p'(u_1) - p'(u_2)}{u_1 - u_2} \right) = \dots$$

$$p(2u) = -2p(u) + \frac{1}{4} \left(\frac{p''(u)}{p'(u)} \right) = -2p(u) + \frac{\frac{1}{4} (6p^2(u) - \frac{1}{2}g_2)^2}{4p^3(u) - g_2p(u) - g_3}$$

$$\Rightarrow p(3u) = p(u + 2u) = \dots$$

$$p(nu) = R(p(u))$$

$$G(f(u), f_1(u)) = 0$$

$$f_1(u) = R_1(p(u; w_1, w_2), p'(u; w_1, w_2))$$

There exists an algebraic dependence between any two elliptic functions w_1, w_2 .

Ex: $\int p^2(u) du$

$$\int p^2(u) du = \int \left(\frac{p''}{6} + \frac{1}{12} g_2 \right) du = \frac{p'}{6} + \frac{1}{2} g_2 u + C$$

$$\int p''^2 du = ?$$

$$p(u) = \frac{1}{u^2} + c_1 u + c_2 u^2 + \dots$$

$\Rightarrow (p'')^2$ has a pole at $u=0$ of order 8.

$$(p'(u))^2 = 4p^3 - g_2 p(u) - g_3$$

$$(p'(u))^2 = \left(-\frac{2}{u^3} + c_1 + 2c_2 u + \dots \right)^2 = \frac{4}{u^6} - 4\frac{c_1}{u^3} + 8\frac{c_2}{u^2} + c_1^2 + 4c_2^2 u^2 + \dots$$

$$= 4 \left(\frac{1}{u^2} + c_1 u + \dots \right)^3 - g_2 \left(\frac{1}{u^2} + c_1 u + \dots \right) - g_3 = \dots$$

$$\Rightarrow c_1 = 0, c_2 = \frac{g_2}{8}, \dots$$

$$\Rightarrow p''(u) = \frac{6}{u^4} + \frac{g_2}{10} + \frac{3}{7} g_3 u^2 + \frac{g_2}{40} u^4 + \dots$$

$$(p'')^2 = \frac{36}{u^8} + \frac{6}{5} g_2 \frac{1}{u^4} + \frac{36}{7} \frac{g_3}{u^2} + \dots$$

but also $(p(u))^{(6)} = \frac{\text{smth}}{u^8} + \dots$ (pole of order 8)

We get an equation

$$\frac{36}{7!} p^{(6)}(u) + \frac{6}{5} \cdot \frac{1}{3!} g_2 p''(u) + \frac{36}{7} g_3 p(u) + A_0 = p''^2$$

How to find A_0 ?

$$\Rightarrow \int (p'')^2 du = c_1 p^{(5)}(u) + c_2 p'(u) + c_3 \int p(u) du + A_0 u + C$$

$$p(au; a\omega_1, a\omega_2) = \frac{1}{a^2} p(u; \omega_1, \omega_2)$$

Ex: Find solution to the equation

$$\left(\frac{dy}{dx}\right)^2 = 4y^3 - g_2 y - g_3$$

$$\int \frac{dy}{\sqrt{4y^3 - g_2 y - g_3}} = \int dx$$

$$I_1 = \int \frac{dx}{\sqrt{P_4(x)}}$$

$$P_4 = a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4$$

how to reduce I_1 to $I_2 = \int \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}$?

in I_1 take $x = c_k + \frac{6p'(c_k)}{24x - P''(c_k)}$

c_k - root of $P_4(x) = P(x)$

$I_3 = \int \frac{dx}{\sqrt{a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3}}$ to reduce to I_2

$$x = -\frac{a_1}{a_0} + \frac{4}{a_0} z$$

$$g_2 = -\frac{a_0}{4} p'(-\frac{a_1}{a_0}), \quad g_3 = \frac{a_0^2}{16} p(-\frac{a_1}{a_0})$$

$$\delta n(u) = \sqrt{\frac{l_1 - l_3}{p(z) - l_3}}$$

Examples of ODE's integrated with $p(u)$ ~~to reduce to~~

• $\left(\frac{dy}{dx}\right)^2 = l(y-a)(y-b)(y-c) \quad y = \frac{4v}{l} + m, \quad m = \frac{a+b+c}{3}$

$$\left(\frac{dv}{dx}\right)^2 = 4v^3 - g_2 v - g_3$$

$v = p(x)$ - some particular solution

the general solution would be $v = p(x-c)$

$$\Rightarrow y = \frac{4}{l} p(x-c) + m$$

• $\left(\frac{dy}{dx}\right)^2 = l(y-a)(y-b)(y-c)(y-d)$, let $y = a + \frac{1}{v}$ then we get

$$y = a + \frac{1}{4} \frac{(a-b)(a-c)(a-d)}{p(x+c) + m l(a-b)(a-c)(a-d)}$$

$$3m = \frac{1}{b-a} + \frac{1}{c-a} + \frac{1}{d-a}$$

$$\left(\frac{dy}{dx}\right)^3 = 4(y-a)^2(y-b)^2$$

$$y = \frac{a+b}{2} + \frac{27}{24} p'(x+c; 0, g_3)$$

$$g_3 = -\frac{(a-b)^2}{729} v^2$$

$$p(u) \text{ - inversion } u = \int_0^\infty \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}$$

$$4x^3 - g_2x - g_3 = 4(x-l_1)(x-l_2)(x-l_3)$$

either all l_i are real or $l_1 \in \mathbb{R}$ $l_2, l_3 = \bar{l}_2 \in \mathbb{C}$

if all real roots assume $l_1 > l_2 > l_3$

$\sqrt{\quad}$ multivalued function

$$\int_\Gamma \frac{1}{\sqrt{4x^3 - g_2x - g_3}} = \int_0^\infty \frac{1}{\sqrt{\quad}} + am + bn, \quad m, n \in \mathbb{Z}$$

$$a = \int_{e_3}^{e_2} \frac{dx}{\sqrt{\quad}} \rightsquigarrow w_1$$

$$b = \int_{e_1}^{e_2} \frac{dx}{\sqrt{\quad}} \rightsquigarrow w_2$$

$$\frac{w}{2} = \int_{e_1}^\infty \frac{dx}{\sqrt{\quad}}; \quad \frac{w'}{2} = i \int_{e_2}^{e_3} \frac{dx}{\sqrt{-4x^3 + g_2x + g_3}}$$

$$p(u + am + nb) = p(u)$$

Ex: $g_2 = 0, g_3 = 1$ Find e_1, e_2, e_3

$$\Rightarrow \text{roots: } \frac{1}{\sqrt[3]{4}}, \frac{1 \pm i\sqrt{3}}{2\sqrt[3]{4}}$$

Degeneration of elliptic functions

$$1) e_1, e_3 = -\frac{1}{2}e_2, \quad g_2 = 3e_1^2, \quad g_3 = e_1^3$$

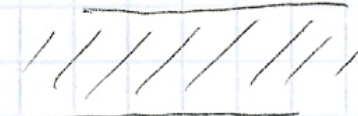
$$g_2^3 - 27g_3^2 = 0$$

$$p(u) = \frac{\left(\frac{\pi}{2\tau}\right)^2}{\sin^2\left(\frac{\pi u}{2\tau}\right)} - \frac{1}{3}\left(\frac{\pi}{2\tau}\right)^2 \quad \text{1-periodic}$$



$$2) e_1 = e_2 = -\frac{e_3}{2}, \quad g_2 = 3e_3^2, \quad g_3 = e_3^3$$

$$p(u) = e_1 + \frac{3e_1}{\operatorname{sh}^2(u\sqrt{3}e_1)}$$



3) if both periods $\rightarrow \infty$

$$\frac{w_1}{w_2} < \infty$$

$$e_1 = e_3 = e_2$$

$$g_2 = 0, \quad g_3 = 0$$

$$\Rightarrow y = \frac{1}{u^2}$$

23.03.2011

$$\zeta'(u) = -P(u)$$

$$\zeta(u) = \frac{1}{u} + \sum_{m, n \neq 0} \left(\frac{1}{u - \tilde{\omega}} + \frac{1}{\tilde{\omega}} + \frac{u}{\tilde{\omega}^2} \right) = \frac{1}{u} - c_2 \frac{u^3}{3} - c_5 u^5 - \dots$$

 $\zeta(u)$ - odd

$$\text{Take } \zeta(u + \omega_k) - \zeta(u) = \eta_k \quad (*)$$

$$\frac{d\eta_k}{du} = \zeta'(u + \omega_k) - \zeta'(u) = 0 \Rightarrow \eta_k - \text{const.}$$

$$\text{Take in } (*) \quad u = -\frac{\omega_k}{2} \Rightarrow \eta_k = 2\zeta\left(\frac{\omega_k}{2}\right) \quad k=1,2$$

$$\Rightarrow \zeta(u + m_1 \omega_1 + m_2 \omega_2) = \zeta(u) + m_1 \eta_1 + m_2 \eta_2, \quad m \in \mathbb{Z}$$

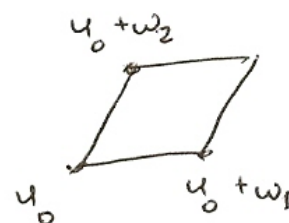
Legendre's relation

$$\text{st. } \eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i$$

Proof: P - for $p(u)$ $\zeta(u)$ has a ^{1-st order} pole at $u=0$

$$\text{Res } \zeta(u) = 1$$

$$\int_{u_0}^{u_0 + \omega_2} (\zeta(u + \omega_1) - \zeta(u)) du - \int_{u_0}^{u_0 + \omega_1} (\zeta(u + \omega_2) - \zeta(u)) du = 2\pi i$$



Expression of elliptic functions by $\zeta(u)$ -function:

$$\zeta(u-a) = \frac{1}{u-a} - c_2 \frac{(u-a)^3}{3} - \dots$$

if $f(u)$ - elliptic function which has in $P(u)$ only simple poles $a_1, \dots, a_r$ with residues $A_1, \dots, A_r$ ($A_1 + \dots + A_r = 0$)

$$\varphi(u) = A_1 \zeta(u-a_1) + \dots + A_r \zeta(u-a_r)$$

$$\zeta(u + \omega_k) - \zeta(u) = \eta_k$$

$$\varphi(u + \omega_k) = \varphi(u) + \sum_i A_i \eta_k = \varphi(u)$$

$$A_1 (\zeta(u-a_1 + \omega_k)) = A_1 (\zeta(u-a_1) + \eta_k)$$

$$f(u) - \varphi(u) \equiv \text{const} \quad (\text{since no poles})$$

what happens to $\varphi(u)$ when we take $\tilde{a}_j \equiv a_j$

$\varphi(u)$ - changes by a constant

Theorem: Let $f(u)$ be an elliptic function with simple

poles at $a_1, \dots, a_r \in P(u_0)$

$$\text{Res}_{u=a_i} f(u) = A_i, \quad i=1, \dots, r$$

$$f(u) = A + A_1 \zeta(u-a_1) + \dots + A_r \zeta(u-a_r)$$

How to express $f(u) = \frac{p'(u)}{p(u)-p(v)}$ in terms of ζ

v - fixed $v \neq \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1 + \omega_2}{2}$ - in order not to have $\frac{0}{0}$

$$\frac{-\frac{2}{u^3}}{\frac{1}{u^2}} = \frac{-2}{u}$$

poles: $u=0, u=v, u=-v$

residues: $\text{Res}_{u=0} f(u) = -2$

$\text{Res}_{u=v} f(u) = 1$

$\text{Res}_{u=-v} f(u) = 1$

$$\sim p(u) - p(v) \sim (u-v)p'(u)$$

$$f(u) = A + \zeta(u+v) + \zeta(u-v) - 2\zeta(u)$$

$$\frac{p'(u)}{p(u) - p(v)}$$

$A = ?$

$u = -u$ and sum up

$A = 0$

Corollary $\frac{1}{2} \frac{p'(u) - p'(v)}{p(u) - p(v)} = \zeta(u-v) - \zeta(u) - \zeta(v)$

Proof: $u = -v$ and sum up

what happens if $\frac{\partial}{\partial u}$ or $\frac{\partial}{\partial v}$ symmetric

$$p(u+v) = p(u) - \frac{1}{2} \frac{\partial}{\partial u} \left(\frac{p'(u) - p'(v)}{p(u) - p(v)} \right) = p(v) - \frac{1}{2} \frac{\partial}{\partial u} \left(\frac{p'(u) - p'(v)}{p(u) - p(v)} \right)$$

How to express $f(u)$ in terms of ζ

principle part

$$f(u) \sim \frac{A}{u-a} - \frac{A_1}{(u-a)^2} + \dots + \frac{(-1)^{k-1}}{(u-a)^k} (k-1)! A_{k-1}$$

$$\varphi(u) = A \zeta(u-a) + A_1 \zeta'(u-a) + \dots + A_{k-1} \zeta^{(k-1)}(u-a)$$

the same p.p. in $u=a$ as in $f(u)$

Theorem:

f - elliptic function

$$f(u) = \text{const} + \sum_{\substack{a \text{ - different} \\ \text{poles of } f(u) \\ \text{in } \mathbb{P}(u_0) \text{ order } k}} \left(A \zeta(u-a) + \dots + A_{k-1} \zeta^{(k-1)}(u-a) \right) =$$

[since $p(u) = -\zeta'(u)$]

$$= c + \sum_a (A \zeta(u-a) - A_1 p(u-a) + \dots + A_{k-1} p^{(k-2)}(u-a))$$

Function $\zeta(u)$: entire function, quasiperiodic

for example e^z - no zeros poles

(rozwiija się w szereg Taylora w każdym punkcie)
one periodic el. f.

$$\zeta(u) - \frac{1}{u} = \sum' \left(\frac{1}{u-\tilde{\omega}} + \frac{1}{\tilde{\omega}} + \frac{u}{\tilde{\omega}^2} \right)$$

$$\int_0^u \zeta(u) - \frac{1}{u} = \sum' \left(\ln\left(1 - \frac{u}{\tilde{\omega}}\right) + \frac{u}{\tilde{\omega}} + \frac{u^2}{2\tilde{\omega}^2} \right) \stackrel{\text{def}}{=} \ln \frac{\sigma(u)}{u}$$

$$\Rightarrow \sigma(u) = u \prod' \left(1 - \frac{u}{\tilde{\omega}} \right) \exp \left(\frac{u}{\tilde{\omega}} + \frac{u^2}{2\tilde{\omega}^2} \right)$$

$\sigma(u)$: • entire

• zeros: $0, \tilde{\omega}$

$$\left(\int_0^u \left(\zeta(t) - \frac{1}{t} \right) dt = \ln \frac{\sigma(u)}{u} \right)'$$

$$\zeta(u) - \frac{1}{u} = \frac{1}{\frac{\sigma(u)}{u}} \cdot \frac{\sigma'(u)u - \sigma(u)}{u^2}$$

$$\Rightarrow \zeta(u) = \frac{\sigma'(u)}{\sigma(u)} = \frac{d \ln \sigma(u)}{du}$$

$$\Rightarrow -p(u) = \frac{\sigma'^2(u) - \sigma''(u)\sigma(u)}{\sigma^2(u)}$$

ζ, p - meromorphic, ratios of entire functions

σ - entire

$$\sigma(u) = u \exp \left\{ -c_2 \frac{u^4}{12} - c_3 \frac{u^6}{30} - \dots \right\} = u + k_2 u^5 + k_3 u^7 + \dots$$

generally k_j - expression in g_2

$\sigma(u)$ - odd

$$\zeta(u + \tilde{\omega}) = \zeta(u) + \eta$$

$$\eta := m_1 \eta_1 + m_2 \eta_2$$

$$\frac{\sigma'(u+\tilde{\omega})}{\sigma(u+\tilde{\omega})} = \frac{\sigma'(u)}{\sigma(u)} + \eta$$

$$\ln \sigma(u+\tilde{\omega}) = \ln \sigma(u) + 2\eta u + \text{const}$$

$$\sigma(u+\tilde{\omega}) = \tilde{c} e^{\eta(\tilde{\omega} + \tilde{\omega}/2)} \cdot \sigma(u)$$

$$\sigma(u+\tilde{\omega}) = \tilde{c} e^{\eta \tilde{\omega}}$$

$$\tilde{c} = e^{\text{const}} e^{-\eta \frac{\tilde{\omega}}{2}}$$

$$u = -\frac{\tilde{\omega}}{2} \Rightarrow \tilde{c} = -1$$

if $\frac{\tilde{\omega}}{2} \in \Lambda$

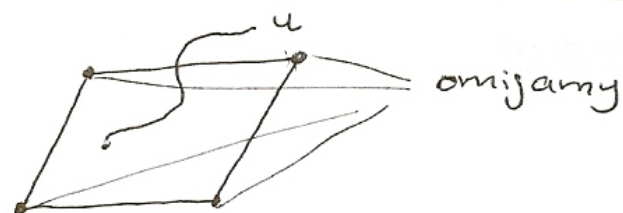
$$\text{then } \sigma\left(\pm \frac{\tilde{\omega}}{2}\right) = 0$$

simple zeros

$$\sigma'\left(\pm \frac{\tilde{\omega}}{2}\right) \neq 0$$

$\sigma'(u)$ - even

$$\tilde{c} = \frac{\sigma'\left(\frac{\tilde{\omega}}{2}\right)}{\sigma'\left(-\frac{\tilde{\omega}}{2}\right)} = 1$$



Theorem:

$$\tilde{\omega} = m_1 \omega_1 + m_2 \omega_2$$

$$\eta = m_1 \eta_1 + m_2 \eta_2$$

$$m_i \in \mathbb{Z}$$

$$\sigma(u + \tilde{\omega}) = \varepsilon \sigma(u) \exp(\eta(u + \frac{\tilde{\omega}}{2}))$$

$$\varepsilon = 1 \quad \text{if} \quad \frac{\tilde{\omega}}{2} \in \Lambda$$

$$= -1 \quad \text{if} \quad \frac{\tilde{\omega}}{2} \notin \Lambda$$

$$\varepsilon = (-1)^{m_1 m_2 + m_1 + m_2}$$

$\sigma(u)$ - is quasi periodic

$$\sigma(u + \omega_1) = -\sigma(u) \exp(\eta_1(u + \frac{\omega_1}{2}))$$

$$\sigma(u + \omega_2) = -\sigma(u) \exp(\eta_2(u + \frac{\omega_2}{2}))$$

$$\left\{ \begin{array}{l} \zeta(u + \omega_k) = \zeta(u) + \eta_k \\ \eta_k = 2\zeta(\frac{\omega_k}{2}), \quad k=1,2 \end{array} \right.$$

if $\varphi(u) := \frac{\sigma(u-b)}{\sigma(u-a)}$ a, b - arbitrary

$$\varphi(u + \tilde{\omega}) = \varphi(u) e^{\eta(a-b)}$$

Task 1: Prove that $\sigma_n(K + iK') = \frac{1}{K}$

σ_n - inverse $\int \frac{ds}{\sqrt{(1-s)^2(1-k^2s^2)}}$ $k \in (0,1)$
 $K' = \sqrt{1-k^2}$

$$K' = \int_0^1 (1-t^2)^{-1/2} (1-k'^2 t^2)^{-1/2} dt$$

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}$$

$$\int_1^{1/k} \frac{ds}{\sqrt{(s^2-1)(1-k^2 s^2)}} \quad s = (1-k'^2 t^2)$$

$\operatorname{Re} k > 0$



* Need to calculate that

$$\int_0^{1/k} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}} = K + iK'$$

$$I = \int_0^{1-\delta} + \int_{1+\delta}^{1/k} + \int_{\gamma}$$

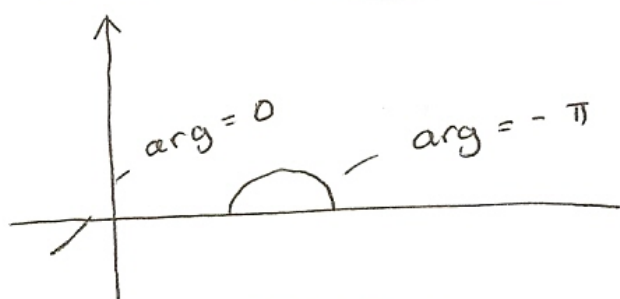
$$\frac{1}{\sqrt{1-t}} \cdot \frac{1}{\sqrt{(1+t)(1-k^2 t^2)}}$$

$$t = 1 + \delta e^{i\varphi}$$

$$\max_t \left| \frac{1}{\sqrt{1-t}} \right|$$

$$\left| \int \frac{1}{\sqrt{(1-t^2)(1-k^2 t^2)}} dt \right| \leq \sup \left| \frac{1}{\sqrt{(1+t)(1-k^2 t^2)}} \right| \left| \int \frac{1}{\sqrt{1-t}} dt \right| \leq$$

$$\leq \text{const} \left| \frac{1}{d^{1/2}} \right| \int_0^{2\pi} 1 dt \rightarrow 0$$



$$t = 1 + \delta$$

$$\sqrt{1-t^2} = \sqrt{-\delta(2+\delta)} = (2+\delta)^{1/2} \delta^{1/2} e^{-\frac{\pi}{2}i} = -i (t^2-1)^{1/2} \Big|_{t=1+\delta}$$

$$(1-k^2 t^2)^{1/2} \Big|_{t=1} \in \mathbb{R}^+$$

$$\int_{1+\delta}^{1/k} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}} = -\frac{1}{c} \underbrace{\int_{1+\delta}^{1/k} -11- dt}_{k'}$$

addition theorem

$$\operatorname{sn}(u + 2k + 2ik') = \operatorname{sn} u$$

sn - odd

$$\operatorname{sn}(u) = u - \frac{1}{6}(1+k^2)u^3 + O(u^5)$$

$$\operatorname{sn}(0) = 0$$

$$\operatorname{sn}(u + ik') = \frac{1}{ku} + \frac{1+k^2}{6k}$$

$$\Rightarrow \operatorname{sn} \text{ has pole at } ik'$$

$$\operatorname{res} = -\frac{1}{k}$$

Ex:

Prove

$$\begin{vmatrix} p(z) & p'(z) & 1 \\ p(y) & p'(y) & 1 \\ p(z+y) & -p'(z+y) & 1 \end{vmatrix} = 0$$

$$p'(y) - A p(y) - B$$

$$s=0 \text{ - pole of 3 order}$$

$$\Rightarrow 3 \text{ zeros}$$

$$\text{chose } p'(z) = A p(z) + B$$

$$p'(y) = A p(y) + B$$

$$0 = z+y + 3\text{rd zero}$$

$$p'(-z-y) = A p(-z-y) + B$$

$$-p'(z+y) \quad p(z+y)$$

$$\begin{vmatrix} -11- & -B & 1 \\ -11- & -B & 1 \\ -11- & -B & 1 \end{vmatrix} = 0$$

Ex: $y = p(z)$ $\frac{3y''}{4(y')^4} - \frac{y'''}{2y'^3} = \frac{3}{16} - \frac{3}{8} \beta$

$$A = \frac{1}{(y-e_1)^2} + \frac{1}{(y-e_2)^2} + \frac{1}{(y-e_3)^2}$$

$$B = y (y-e_1)^{-1} (y-e_2)^{-1} (y-e_3)^{-1}$$

$$(y')^2 = 4(y-e_1)(y-e_2)(y-e_3)$$

$$\frac{3(y'')^2}{4(y')^4} = \frac{3/4 \left(\frac{(\log y')'}{y'} \right)^2}{1} = \frac{3}{4} \left(\frac{12 \sum \log y-e_k}{y'} \right)^2 =$$

$$= \frac{3}{4} \left(\frac{\sum \frac{1}{y-e_k}}{y'} \right)^2 = \frac{3}{4} \frac{(\sum \frac{1}{y-e_j})^2}{4\pi(y-e_i)} =$$

$$= \frac{3}{16} \frac{1}{\pi y-e_i} \sum_i \frac{1}{(y-e_j)^2} + \sum_{i,j} \frac{2}{(y-e_j)(y-e_i)} =$$

$$\left(\frac{2y''}{(y')^2} \right)' = \frac{2y'''}{y'^3} - \frac{4y''^2}{y'^4}$$

Ex:

$$f'' = 6f^2 - 4f$$

$$2pp'' = 12p^2p' - g_2$$

$$2f'f'' = 6f^2 - 4f$$

$$(f')^2 = 4f^2(f-1) + \text{const}$$

sol: $f = \frac{1}{\cos^2 t}$

$$f' = \frac{2 \sin t}{(\cos^2 t)^2}$$

$$\int_{f(2)}^{\infty} \frac{1}{t \sqrt{t-1}} dt$$

classical function

Expression of elliptic functions in terms of $\sigma(u)$

$f(u)$ - elliptic function of order r with periods ω_1, ω_2

$\{b_1, \dots, b_r\}$ - r zeros in $\mathbb{P}$

$$a_1 + \dots + a_r \equiv b_1 + \dots + b_r$$

$\{a_1, \dots, a_r\}$ - r poles in $\mathbb{P}$

(we can always choose b_i, a_i s.t.
 $a_1 + \dots + a_r \equiv b_1 + \dots + b_r$)

$$F(u) = \frac{\sigma(u-b_1) \dots \sigma(u-b_r)}{\sigma(u-a_1) \dots \sigma(u-a_r)}$$

$$F(u + \tilde{\omega}) = F(u) \exp \left\{ \underbrace{\gamma \{ (a_1 - b_1) + \dots + (a_r - b_r) \}}_0 \right\} = F(u)$$

$$\frac{F(u)}{f(u)} = \text{const}$$

$$f(u) = \text{const} \frac{\sigma(u-b_1) \dots \sigma(u-b_r)}{\sigma(u-a_1) \dots \sigma(u-a_r)} \quad a_1 + \dots + a_r = b_1 + \dots + b_r \quad !$$

Exercise:

$$f(u) = p(u) - p(v)$$

v - any fixed number not equal to $\tilde{\omega} = m_1 \omega_1 + m_2 \omega_2$

$p(u)$ one pole in 0 double order

$f(u)$ - order 2, elliptic

$$a_1 = a_2 = 0$$

$$b_1 = v$$

$$b_2 = -v$$

$$\Rightarrow f(u) = \text{const} \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)} = -\frac{1}{\sigma^2(v)} \cdot \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)}$$

functions $p(u), \xi(u), \sigma(u)$ as also functions of ω_1, ω_2

$$\text{Im} \frac{\omega_2}{\omega_1} > 0$$

$$p(u, \omega_1, \omega_2) = \frac{1}{u^2} + \sum'_{\eta_1, \eta_2 \neq 0} \left(\frac{1}{(u - \tilde{\omega})^2} - \frac{1}{\tilde{\omega}^2} \right), \quad \tilde{\omega} = \eta_1 \omega_1 + \eta_2 \omega_2$$

$$p(\lambda u, \lambda \omega_1, \lambda \omega_2) = \lambda^{-2} p(u, \omega_1, \omega_2)$$

$$\xi(\lambda u, \lambda \omega_1, \lambda \omega_2) = \lambda^{-1} \xi(u, \omega_1, \omega_2)$$

$$\sigma(\lambda u, \lambda \omega_1, \lambda \omega_2) = \lambda \sigma(u, \omega_1, \omega_2)$$

$$p(u, \omega_1, \omega_2) = \omega_2^2 p\left(\frac{u}{\omega_1}, 1, \frac{\omega_2}{\omega_1}\right)$$

$$\xi(u, \omega_1, \omega_2) = \omega_1 \xi\left(\frac{u}{\omega_1}, 1, \frac{\omega_2}{\omega_1}\right)$$

$$\sigma(u, \omega_1, \omega_2) = \frac{1}{\omega_1} \sigma\left(\frac{u}{\omega_1}, 1, \frac{\omega_2}{\omega_1}\right)$$

$$p(u, \omega_1, \omega_2) = p(u, \omega_1', \omega_2')$$

$\Leftrightarrow$ when their poles coincide

$$\{m_1 \omega_1 + m_2 \omega_2\}$$

$$\{m_1' \omega_1' + m_2' \omega_2'\}$$

Theorem: $\text{Im} \frac{\omega_2}{\omega_1} \neq 0$

functions coincide $\Leftrightarrow$

$$\omega_2' = \alpha \omega_2 + \beta \omega_1$$

$$\omega_1' = \gamma \omega_2 + \delta \omega_1$$

$$\alpha, \beta, \gamma, \delta \in \mathbb{Z}$$

$$\alpha \delta - \beta \gamma = \pm 1 \quad \Leftrightarrow \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \pm 1$$

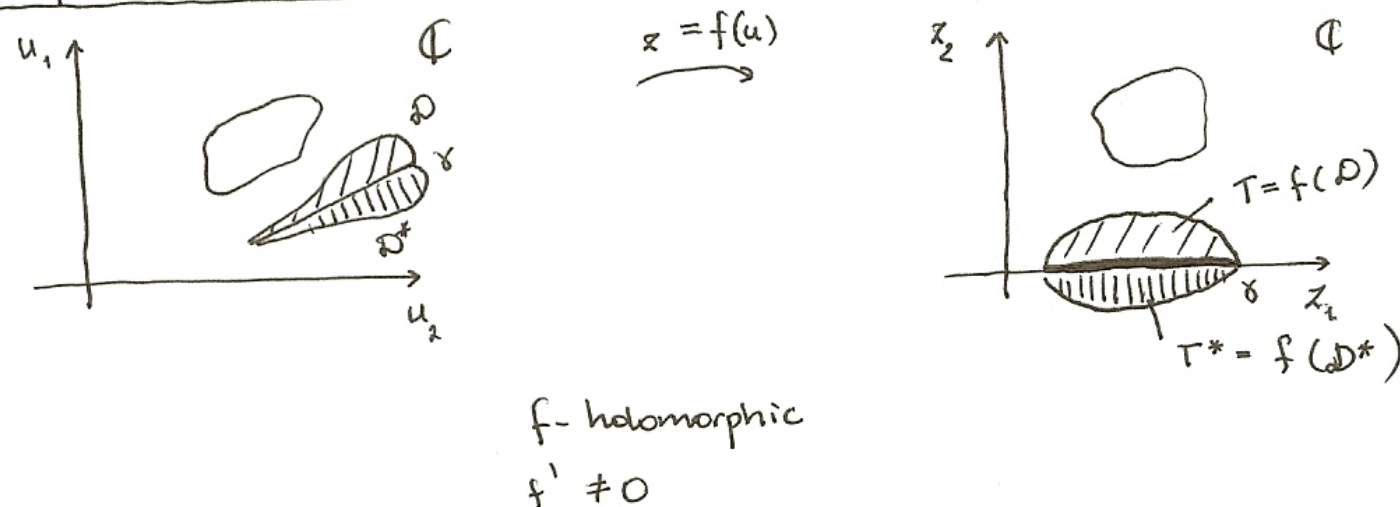
$$\begin{pmatrix} w_2' \\ w_1' \end{pmatrix} = A \begin{pmatrix} w_2 \\ w_1 \end{pmatrix} \quad \begin{pmatrix} w_2 \\ w_1 \end{pmatrix} = A^{-1} \begin{pmatrix} w_2' \\ w_1' \end{pmatrix}$$

Note $g_2 = 60 \sum' \frac{1}{\omega^4}$
 $g_3 = 140 \sum' \frac{1}{\omega^4}$

$$(p'(u))^2 = 4p^3 - g_2 p(u) - g_3$$

g_2, g_3 are invariants with respect to modular transformation

conformal transformations using elliptic functions



symmetry principle

$f(u)$ - symmetric function of order r

zeros in $\mathbb{P}$

$$u \in \mathbb{C}/\Lambda \xrightarrow{f} \mathbb{C} \ni z \quad \text{not } 1:1$$

$$\mathbb{C}/\tilde{\Lambda} \subset \mathbb{P} \xrightarrow{f} \mathbb{C} \quad 1:1$$

g_2 on was inverse to

$$u = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$0 < k < 1$$

$$k': k^2 + k'^2 = 1$$

$$z = \text{sn}(u, k)$$

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$K' = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}}$$

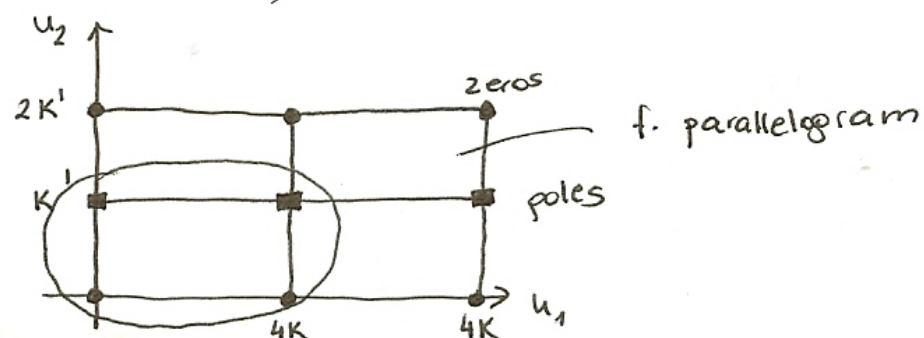
$$\text{sn}(u + 4mK + 2niK') = \text{sn}(u)$$

$$\begin{aligned} w_1 &= 4K \\ w_2 &= 2K'i \end{aligned} > \text{periods of } \text{sn}(u)$$

$$\text{sn}(u + 2K + 2iK') = -\text{sn}(u) \quad (\text{last week})$$

$$\text{zeros of sn: } u = 2Km + 2nK'i, \quad m, n \in \mathbb{Z}$$

$$\text{poles (1st order)} \quad u = 2mk + (2n+1)K'i, \quad \text{residues } \frac{(-1)^m}{k}$$



$$\int_0^1 \frac{dx}{\sqrt{x}} = K \quad \int_0^{1/k} \frac{dx}{\sqrt{x}} = K + iK' \quad (\text{evaluated in detail last time})$$

$$z = \operatorname{sn}(u_1 + iu_2)$$

$$u_1 = \operatorname{Re} u$$

$$u_2 = \operatorname{Im} u$$

values of sn in $\mathbb{R}$:

$u_2 \backslash u_1$	0	K	2K	3K
0	0	1	0	-1
K'	∞	$1/k$	∞	$-1/k$

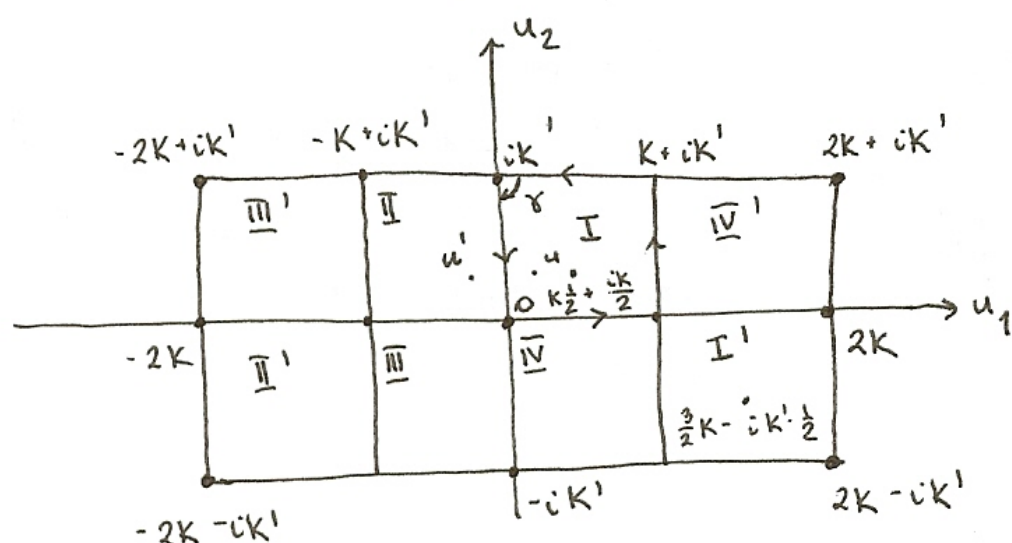
$$\operatorname{sn}(-u) = -\operatorname{sn}(u)$$

$$k \in \mathbb{R}, k \in (-1, 1) \quad \operatorname{sn}(u) \in [-1, 1]$$

$\operatorname{sn}(u \pm v)$ = addition formulas

values of $z = \operatorname{sn}(u)$

$$\operatorname{sn}(iu, k) = i \frac{\operatorname{sn}(u, k')}{\operatorname{cn}(u, -k')}$$

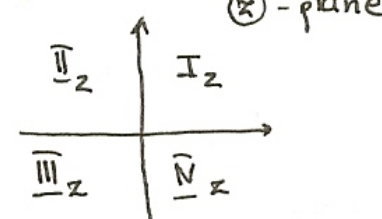
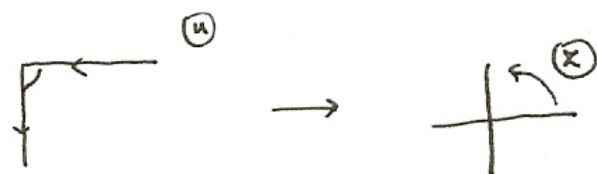


$$\mathbb{P}(-2K - iK') = \{u \in \mathbb{C} \mid u = 2Kt + t'K'i, -1 \leq t < 1, -1 \leq t' < 1\}$$

I if $u \in [0, K]$
 $\Rightarrow z = x_1 + ix_2$ changes from 0 to 1
 if u changes from K to $K + iK' \Rightarrow z$ changes from 1 to $1/k$
 $K + iK' \rightarrow K'i$
 $K'i \rightarrow 0$
 $1/k \rightarrow \infty$
 $\infty \rightarrow 0$ (along im. axis)

If in u plane γ -small circle around $K'i \Rightarrow$ in z plane we have the image of γ denoted $\tilde{\gamma}$ "close to ∞ " distant from the origin

in u -plane as going from $K + K'i$ to iK' and then go along imaginary axis
 in z -plane we also switch from real to imaginary axis



I in $(u) \mapsto I_z$ (first quadrant of z plane)

By symmetry w.r.t. imaginary axis in u plane the points in II go to $II_z \in (\mathbb{Z})$ -plane

$$III, IV \mapsto III_z, IV_z$$

In I', II', III', IV' the function sn has the same values as in I, II, III, IV respectively.

$$\ln\left(\frac{k}{2} + \frac{ik'}{2}\right)$$

$$\operatorname{sn}\left(\frac{3}{2}K - i\frac{K'}{2}\right) = \operatorname{sn}\left(-\left(\frac{K}{2} + \frac{ik'}{2}\right) + 2K\right)$$

I u II $\xrightarrow{\operatorname{sn}}$ upper half z -plane

z -plane $\operatorname{Re} z > 0 \xrightarrow{\text{conformally}}$

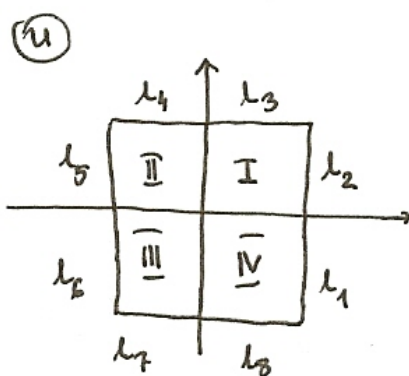
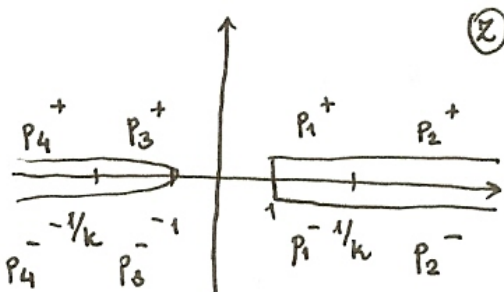


$$u = \int_0^z \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

Corollary: There is a conformal transformation from a circle into rectangular $K + K'i, -K + K'i, -K - K'i, K - K'i \rightarrow z$ plane

z plane is cut along $(-\infty, -1) \cup (1, \infty)$

what is the image of upper/lower part of the cut in u plane



p_1^+ - upper cut of z plane from 1 to $\frac{1}{k}$ corresponds to l_2 (side from K to $K + iK'$)

$$p_1^- \Leftrightarrow l_1$$

p_2^+ (cut from $(\frac{1}{k}, \infty)$) l_3 (side $(K + K'i, K'i)$)

p_2^- l_1 (side $(K - K'i, -K'i)$)

right cut in z plane $\Rightarrow$ right boundary

$$\operatorname{Im} \frac{w_2}{w_1} \neq 0$$

$\leadsto$ el. functions

$$\frac{w_2}{w_1} \in \mathbb{Q} \quad 1\text{-period functions}$$

$$\frac{w_1}{w_2} \in \mathbb{Q} \setminus \mathbb{R} \quad \text{constant}$$

Weierstrass identity for σ function

$$\sigma(z+a)\sigma(z-a)\sigma(b+c)\sigma(b-c) + \sigma(z+b)\sigma(z-b)\sigma(c+a)\sigma(c-a) + \sigma(z+c)\sigma(z-c)\sigma(a+b)\sigma(a-b) = 0$$

Halphen proved that the only function which satisfies this discrete equation is $\sigma(z)$

uniformization of curves (of genus 1):
elliptic curves

y - 2-valued function
 x - 4-valued function

$$y^2 = a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4 = f(x)$$

$$a_0, a_1 \neq 0$$

x, y can be expressed as single-valued functions of some auxiliary variable z

$x = x(z)$ > single valued

$y = y(z)$

$$x = x_0 + \frac{1}{4} f'(x_0) \left\{ p(z) - \frac{1}{24} f''(x_0) \right\}^{-1}$$

$$y = -\frac{1}{4} f'(x_0) p'(z) \{ p(z) - \frac{1}{24} f''(x_0) \}^{-2}$$

$$f(x_0) = 0$$

$$z = \int_{x_0}^x \{ f(t) \}^{-1/2} dt$$

$$(p')^2 = 4p^3(z) - g_2 p(z) - g_3$$

$$g_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2$$

$$g_3 = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 - a_1^2 a_4$$

Sketch

$$f(t) = 4A_3(t-x_0) + 6A_2(t-x_0)^2 + 4A_1(t-x_0)^3 + A_0(t-x_0)^4$$

$$f(x_0) = 0$$

$$A_0 = a_0$$

$$A_2 = a_0 x_0^2 + 2a_1 x_0 + a_2$$

$$(t-x_0)^{-1} = \tau$$

$$A_1 = a_0 x_0 + a_1$$

$$A_3 = a_0 x_0^3 + 3a_1 x_0^2 + 3a_2 x_0 + a_3$$

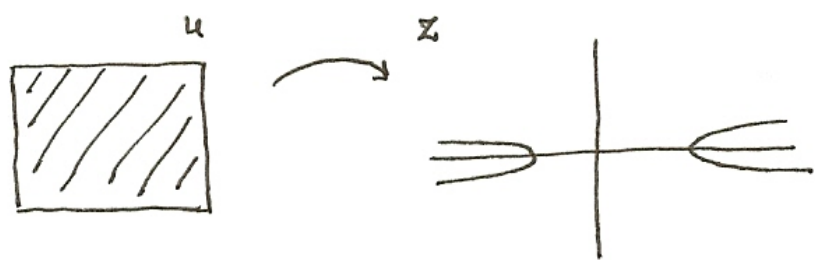
$$(x-x_0)^{-1} = \xi$$

$$z = \int_{\xi}^{\infty} \{ 4A_3 \tau^3 + 6A_2 \tau^2 + 4A_1 \tau + A_0 \} d\tau$$

$$\tau = A_3^{-1} (\sigma - \frac{1}{2} A_2)$$

$$\xi = A_3^{-1} (s - \frac{1}{2} A_2)$$

$$z = \int_s^{\infty} (4\sigma^3 - (3A_2^2 - 4A_1 A_3)\sigma - (2A_1 A_2 A_3 - A_2^3 - A_0 A_3^2)) d\sigma$$



$$\text{div}(f) = \sum \text{ord}_a(f)(a) = \sum_{\substack{n_i \in \mathbb{Z} \\ \text{point}}} n_i \cdot (a)$$

$$\text{div}(p') = -3(0) + 1 \cdot \left(\frac{w_1}{2}\right) + 1 \cdot \left(\frac{w_1 + w_2}{2}\right)$$

$$\sum n_i(a) \equiv 0$$

$$\text{div}(p(u) - p(v)) = -2(0) + 1 \cdot (v) + 1 \cdot (-v)$$

$$\text{div}(p - e_1) = -2(0) + 2 \cdot \left(\frac{w_1}{2}\right)$$

06.04.2011

Riemann surfaces:

$$f(z) = \sqrt{z}$$

$$f(0) = 0$$

$$z = r e^{i\varphi}$$

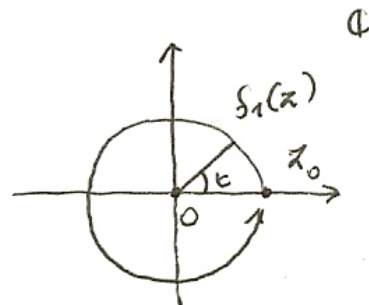
$$\varphi \in [0, 2\pi]$$

$$f_1(z) = \sqrt{r} e^{i\frac{\varphi}{2}}$$

$$f_2(z) = -\sqrt{r} e^{i\frac{\varphi}{2}}$$

$$z \mapsto (f_1(z), f_2(z)) \in \mathbb{C} \times \mathbb{C} \quad \text{multivaluedness}$$

$$\forall z \in \mathbb{C} \setminus \{0\}$$



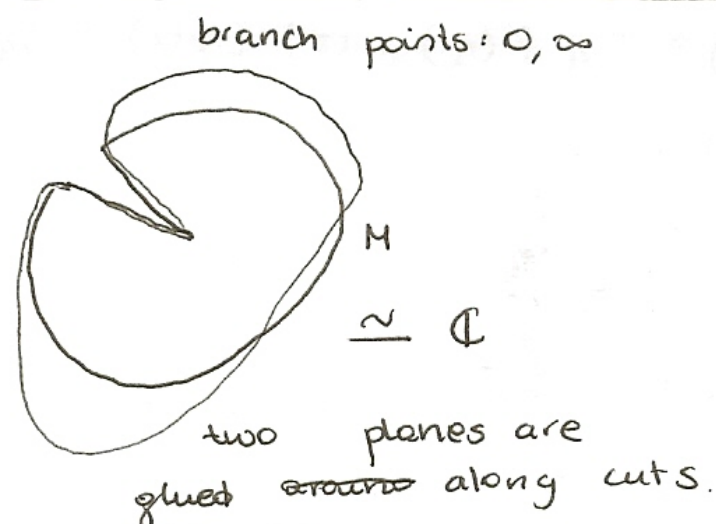
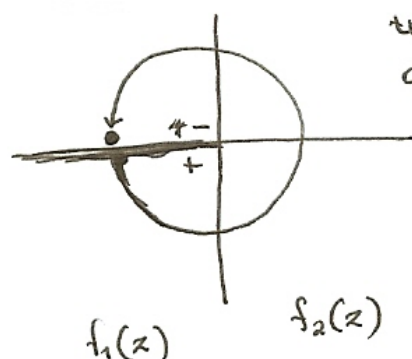
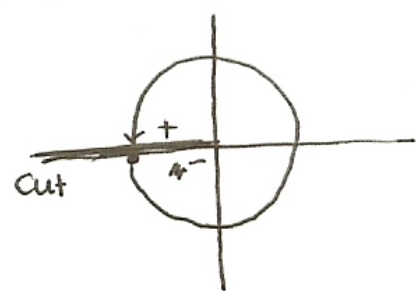
$$f_1(e^{it\frac{2\pi}{n}} z) = \sqrt{r} e^{i\frac{(t+\varphi)}{2}} = f_1(z) e^{i\frac{t}{2}}$$

$$f_2(e^{it\frac{2\pi}{n}} z) = e^{it/2} f_2(z)$$

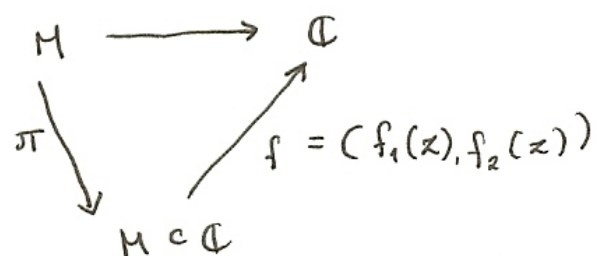
$$f_1(e^{2\pi i} z) = -f_1(z) = f_2(z)$$

$$f_2(e^{2\pi i} z) = f_1(z)$$

interchange of branches after a loop around 0



Def: Riemann surface for multivalued f is manifold M



$\pi^{-1}(M)$ - countable

$$w = f = \sqrt{z}$$

$$M \cong \mathbb{C}$$

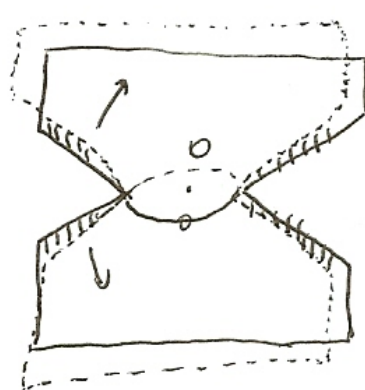
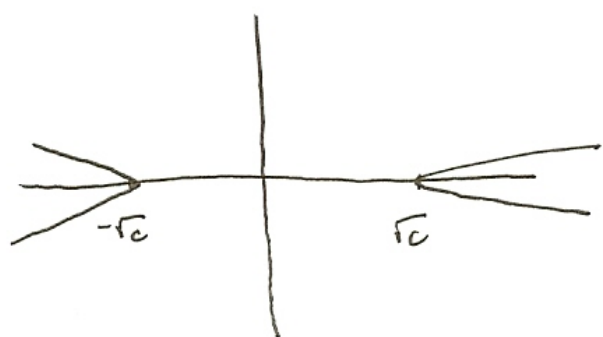
$$\pi: w \rightarrow w^2 = z$$

$$\tilde{f}(w) = w$$

$$f = \tilde{f} \circ \pi^{-1} = \pm \sqrt{z}$$

Ex:

$$w = \sqrt{c - z^2}, \quad c > 0$$



$\cong$

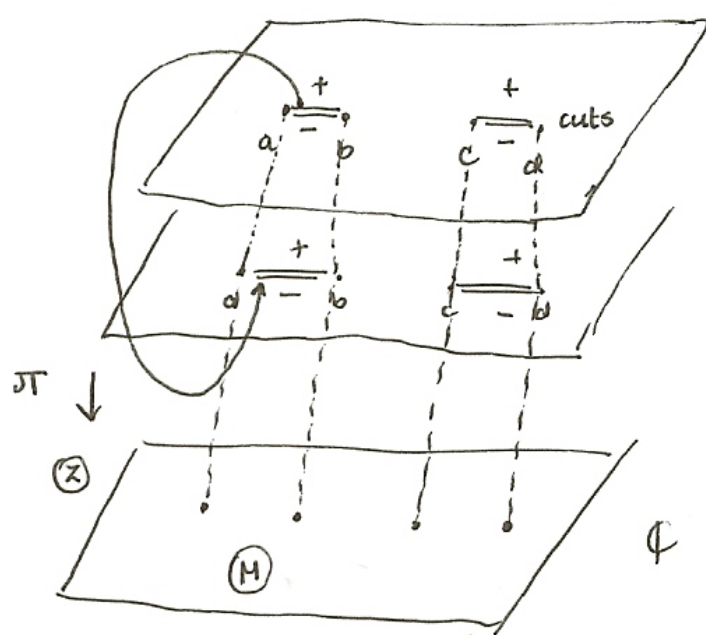


branch points: $-\sqrt{c}, \sqrt{c}, \infty$

Ex:

$$y^2 = (1-x)(1+x)(1-ix)(1+ix)$$

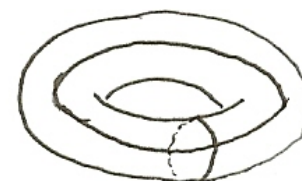
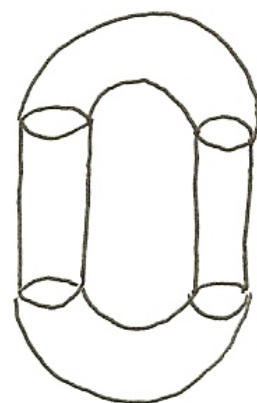
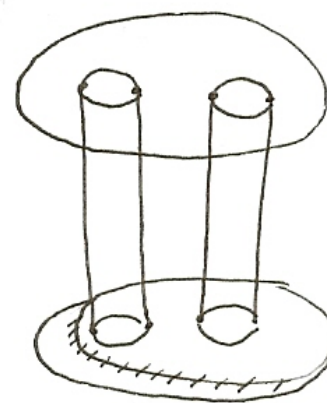
finite branch points: $1, -1, i, -i$



$\mathbb{C}$

images of $f(z)$

diffeo



torus
genus 1
Riemann surface
2 cycles

Def: function f is meromorphic on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ (Riemann's sphere)

if $\forall a \in \mathbb{C} \exists k \in \mathbb{N}$ st. $(z-a)^k f(z)$ is holomorphic at a

Ex:

$$f = \frac{1}{z} \quad z \cdot f(z) = 1$$

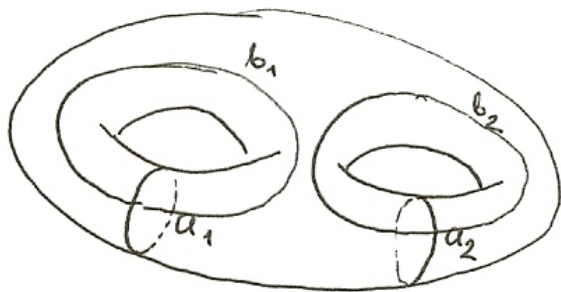
$$\text{if } a = \infty \Rightarrow z \rightarrow \frac{1}{z} = \frac{1}{\infty} = 0 \quad \infty \rightarrow 0$$

Prop The functions meromorphic on $\mathbb{P}_1(\mathbb{C})$ are rational functions, because
 The sum of res on $\mathbb{P}_1(\mathbb{C}) = 0$

Def: For genus 1 Riemann's surface $\exists$ holomorphic differentials
 $\frac{dx}{y^i}, i=1$

Def: $y^2 = P_g(x)$

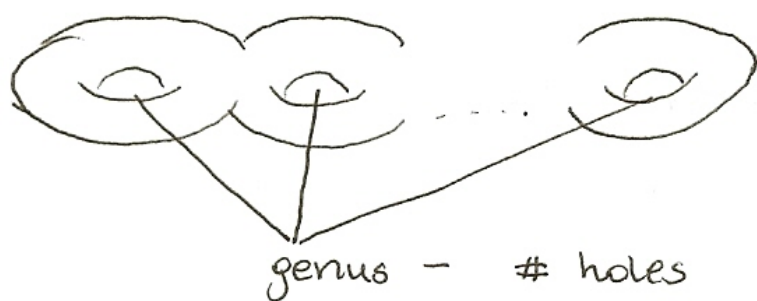
genus 2 = g



matrice of periods $\int_{\gamma_k} \frac{dx}{y^i}$

Remark: Riemann surface is always compact

$$y = \sqrt{P_n(z)}$$



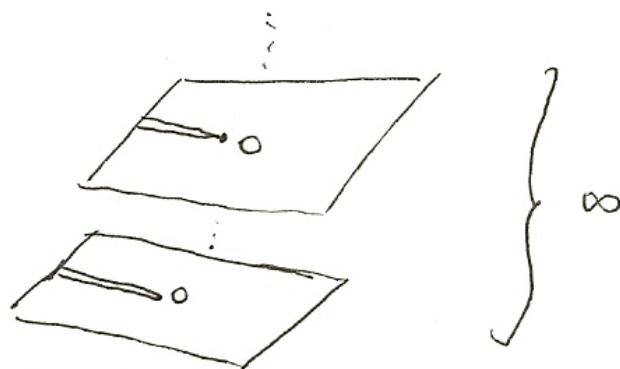
genus - # holes

Example: $\log$

$$z = re^{i\varphi}$$

$$\log z = \log|z| + i\arg z + 2\pi i k$$

$$B = \mathbb{C} \setminus \{0\}$$



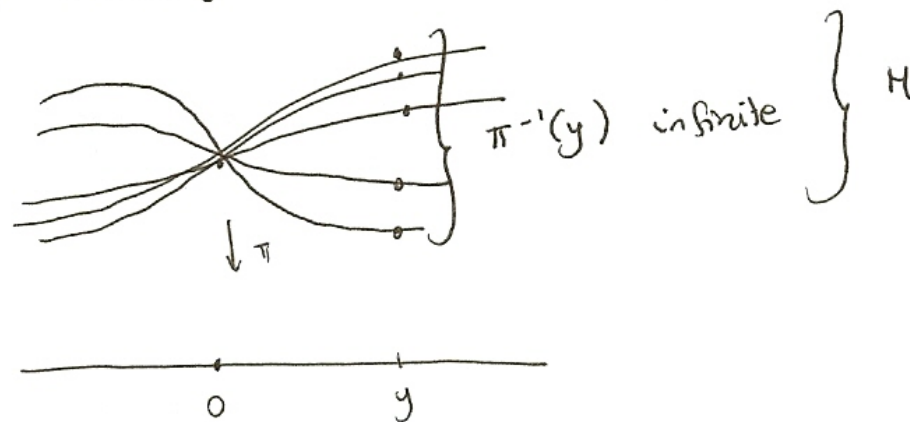
$M =$ "infinite spiral of planes"

$$\pi^{-1}(y) = \{x \in M : e^x = y\} = \{\ln y + 2\pi i k, k \in \mathbb{Z}\}$$

$$\pi : x \rightarrow e^x$$

$$M = \bigsqcup_{k \in \mathbb{Z}} \mathbb{C}^*$$

$$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$$



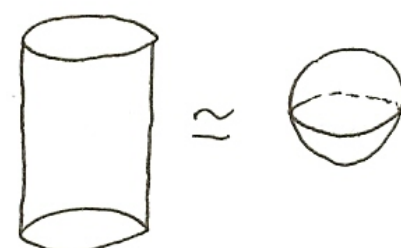
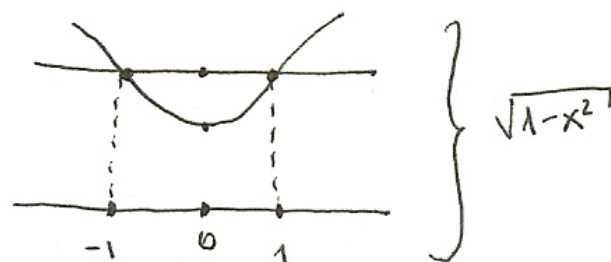
Remark:

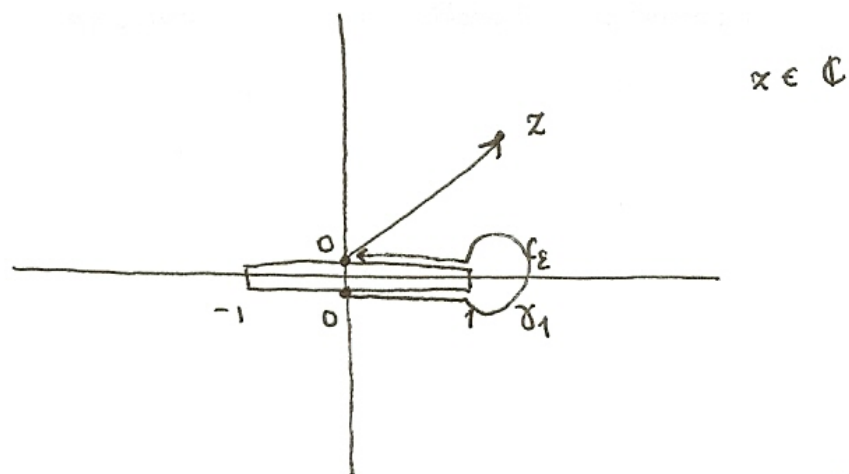
for $\sqrt{z} \rightarrow$ genus 0

for $\sqrt{c-z^2} \rightarrow$ genus 0

Example:

$$z \rightarrow \int_0^z \frac{dx}{\sqrt{1-x^2}} = I''(z)$$





$$I(z) \xrightarrow{\gamma_1} ? \quad \int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}} + \int_{C_\varepsilon} \frac{dx}{\sqrt{1-x^2}} + \int_{1-\varepsilon}^0 -\frac{dx}{\sqrt{1-x^2}} \quad \begin{array}{l} \xrightarrow{\varepsilon \rightarrow 0} 0 \\ \text{because the sign} \\ \text{changes on the other} \\ \text{side of cuts} \end{array} \neq -I(z)$$

C_ε - small circle around 1

$$I(z) \xrightarrow{\gamma_1} -I(z) + 2 \int_0^1 \frac{dx}{\sqrt{1-x^2}} = -I(z) + 2\pi$$

$$C_\varepsilon: x = 1 + \varepsilon e^{i\varphi} \quad \left| \int \frac{dx}{\sqrt{(1-x)(1+x)}} \right| \leq C \int \frac{d\varphi \varepsilon}{\sqrt{\varepsilon}} = \sqrt{\varepsilon} \cdot C \xrightarrow{\varepsilon \rightarrow 0} 0$$

$$I(x) \rightarrow \pm I(x) + \pi k \quad - \text{depends on chosen loops}$$

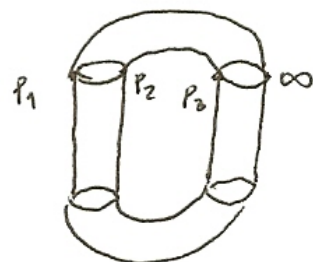
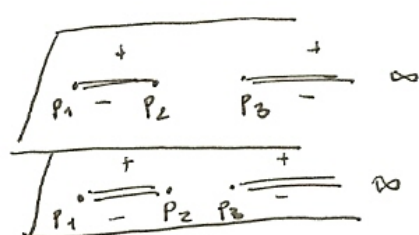
$$\sin(x + \pi k) = \pm \sin x$$

Example:

$$\{y^2 = 4x^3 - g_2x + g_3\} \subset \mathbb{C}^2$$

$$p(x) = 4(x-e_1)(x-e_2)(x-e_3) \quad \sum e_i = 0 \quad (\text{no } x^2 \text{ term})$$

$$Pe_1 = (e_1, 0), \quad Pe_2 = (e_2, 0), \quad Pe_3 = (e_3, 0)$$



$$p(\infty, \infty) \quad y = \frac{1}{y'}, \quad x = \frac{1}{x'} \quad - \text{ramification point}$$

$$y' = \pm \varepsilon x'^{3/2} \left(\sqrt{(1-e_1x')(1-e_2x')(1-e_3x')} \right)^{-1}$$

Exercise:

$$I(P) = \int_{P_0}^P \frac{dx}{y}$$

$$\frac{dx}{y} \quad - \text{holomorphic differential?}$$

• if $P \neq Pe_i, \infty$

$$y = \sqrt{4(x-e_1)(x-e_2)(x-e_3)} \quad \text{holomorphic}$$

• $Pe_i: f(x, y) = y^2 - p_3(x)$

$$\left. \frac{\partial f}{\partial y} \right|_{Pe_i} = 0 \quad \left. \frac{\partial f}{\partial x} \right|_{Pe_i} \neq 0 \quad \Rightarrow x - e_i = t_i^2 \quad - \text{locally}$$

$$dx = 2t_i dt_i$$

$$y^2 = 4t_i^2 (t_i + \dots)(t_i + \dots)$$

$$y = \pm 2t_i \sqrt{Q(t_i)}$$

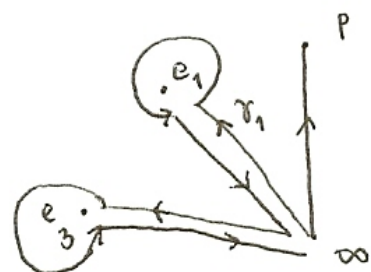
$$\text{holomorphic at } t_i = 0$$

• $\infty: x = \frac{1}{t_\infty^2} \text{ - loc} \quad dx = -2t_\infty^{-3} dt_\infty \quad -42-$

$$y^2 = \frac{4(1-e_1 t_\infty^2)(1-e_2 t_\infty^2)(1-e_3 t_\infty^2)}{t_\infty^6}$$

$$y = \pm \frac{1}{t_\infty^3} Q(t_\infty) \quad \text{holomorphic}$$

$$\frac{dx}{y} - \text{holo}$$

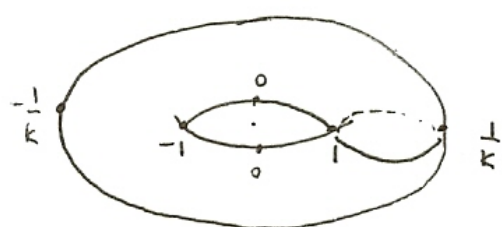


$$I(P) = \int_{\infty}^P \frac{dx}{\sqrt{p_3(x)}} = \gamma_1 = \int_{\infty}^{e_1} + \int_{\text{loop around } e_1} - \int_{e_1}^{\infty} = I(P)$$

$$w_1 = 2 \int_{\infty}^{e_1}$$

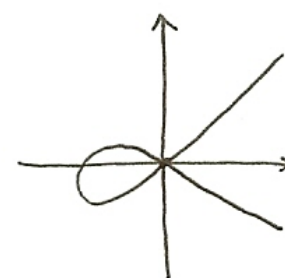
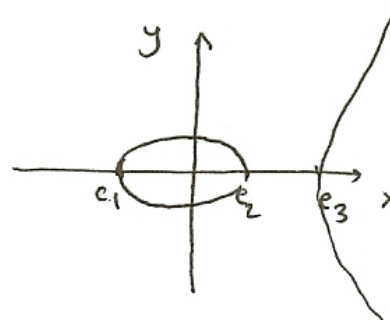
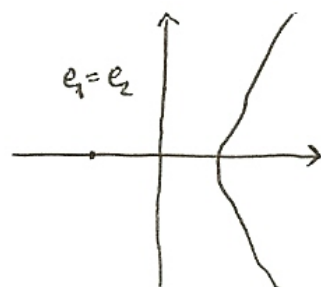
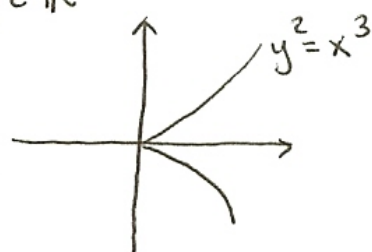
$$I(P) - w_1, w_2 \quad \text{periodic}$$

$$w_2 = 2 \int_{\infty}^{e_3} \frac{dx}{y}$$



$$y^2 = p_3(x)$$

$$x, y \in \mathbb{R}$$



How to calculate genus on examples using Riemann's - Roch theorem

Example:

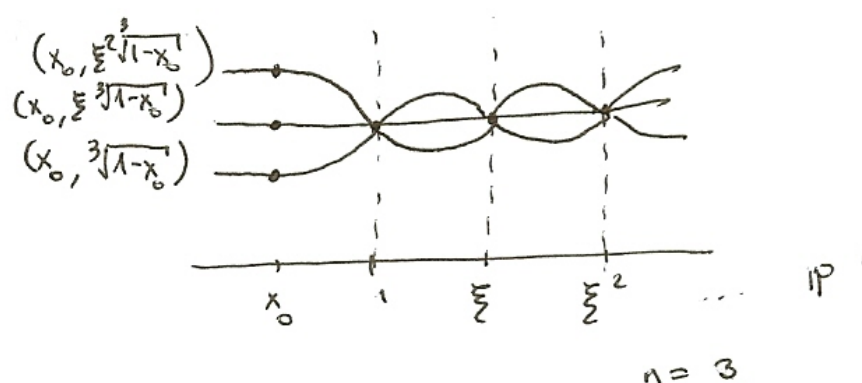
$$x^n + y^n = 1 \quad \text{Fermat's curve}$$

$$y^n = 1 - x^n = (1-x)(\xi - x)(\xi^{n-1} - x)$$

$$\xi = e^{2\pi i/n}$$

$$p_k = (\xi^k, 0) \quad k \in \{0, \dots, n-1\}$$

$\{p_0, \dots, p_{n-1}\}$ - ramification points



$$\text{in } p_k \in S$$

$$X = x - \xi^k$$

$$y^n = X \varphi(X), \quad \varphi(\xi^k) \neq 0$$

$$Y^n = X \quad (x, y) \mapsto y^{\odot} - \text{index of ramification number}$$

$$\begin{cases} f(x_0, y_0) = 0 \\ \frac{\partial f}{\partial y}(x_0, y_0) = 0 \end{cases}$$

$\Rightarrow (x_0, y_0)$ - ramification point

Remark:

$$f_n(x, y) = 0$$

in general the ramification picture is complicated

Riemann's - Roch's formula (has to calculate genus)

$$X \xrightarrow{f} Y \quad \text{- alg of } (x, y) \quad \mathbb{P}_1(\mathbb{C})$$

$$g_Y = 0$$

$$2g_X - 2 = \underbrace{-2n}_{\deg_y f} + \sum_{p \in X} \underbrace{(e_f(p) - 1)}_{\substack{\text{order} \\ \text{ramification index}}}$$

Subexample:

1) $x = y^q$
 $X = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid x = y^q\} \xrightarrow{\pi(x, y)} \mathbb{C} \ni x$

$$2g_X - 2 = -2q + \underbrace{(q-1)}_{p=0} + \underbrace{(q-1)}_{p=\infty}$$

$g_X = 0 \leadsto$ Riemann's sphere

2) $X: y^2 = (x-a_1) \dots (x-a_n) \subset \mathbb{C}^2$

if ∞ is not ramification point

$$2g_X - 2 = -4 + n(2-1)$$

$$g_X = \frac{n-2}{2}$$

How to check if ∞ is ramification point? $\nRightarrow n \neq 2k$

$$y = \frac{1}{y'}, \quad x = \frac{1}{x'}$$

$$\frac{1}{y'^2} = \frac{(1-a_1 x') \dots (1-a_n x')}{x'^n}$$

$$y'^2 = x'^n \cdot \frac{1}{(1-a_1 x') \dots (1-a_n x')}$$

$n = 2k$ - ok $\Rightarrow \infty$ is not ram. point $\Leftrightarrow n = 2k$

$n = 2k+1$ - problem

$$2g_X - 2 = -4n + n + 1$$

$$g = \frac{n-1}{2}$$

3) calculate genus for Fermat's curve

$$x^n + y^n = 1$$

$$2g_X - 2 = -2n + n(n-1) \quad g_X = \frac{(n-1)(n-2)}{2}$$

∞ is not ramification point: $y^n = \prod_{k=0}^{n-1} (x - \xi^k)$, $\frac{1}{y^n} = \frac{\prod_{k=0}^{n-1} (1 - \xi^k x)}{x^n}$

Example:

$$f(z, w) = w^4 - 2w^2 + 1 - z = 0$$

Calculate genus

$$w(z) = \sqrt{1 + \sqrt{z}}$$

ramification point: $f(z_0, w_0) = 0$

$$\frac{\partial f}{\partial w}(z_0, w_0) = 0$$

$$\frac{\partial f}{\partial w} = 4w^3 - 4w = 0 \Rightarrow w = -1, 0, 1$$

$$z = 0 \quad z = 1$$

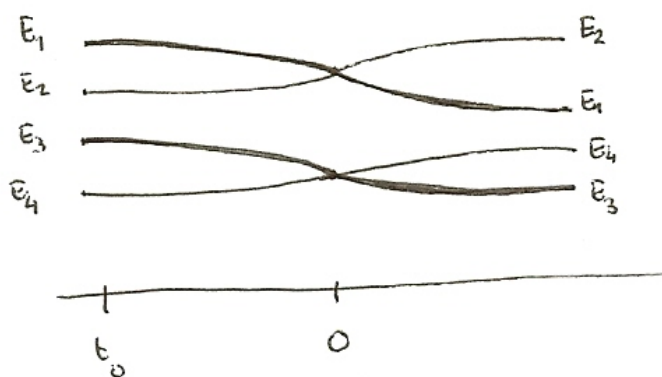
$$(z_0, w_0) = \{(0, \pm 1), (1, 0)\}$$

$$f_1(\alpha) = \sqrt{1+\alpha}$$

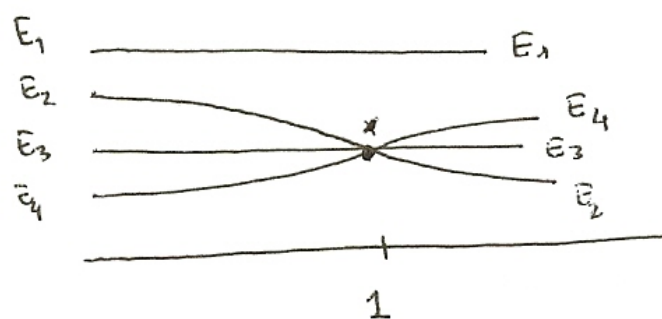
$$f_2(\alpha) = \sqrt{1-\alpha}$$

$$f_3(\alpha) = -\sqrt{1+\alpha}$$

$$f_4(\alpha) = -\sqrt{1-\alpha}$$



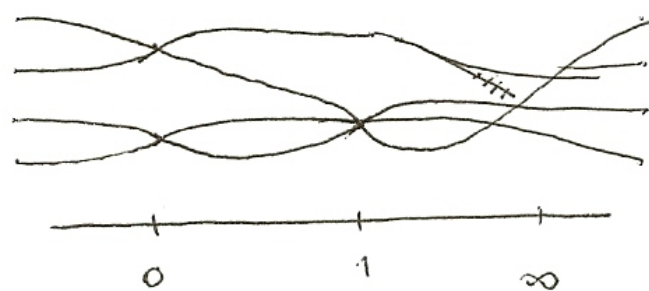
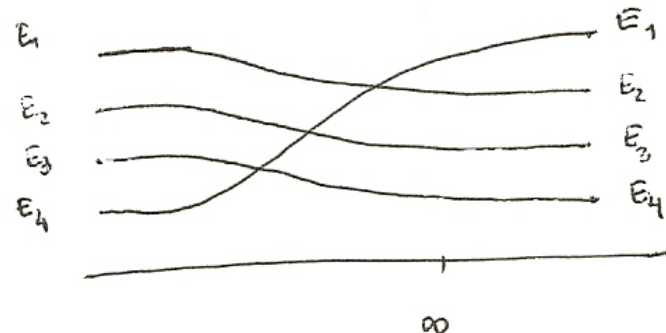
Improper



Proper:



? here might be a mistake



2 values for $\xi = \sqrt{z}$

2 values for $f(\xi) = \sqrt{1+\xi}$

$$\deg_w f = 4$$

2 ramification points of index 2 (order 1) at $z = 0$

1 ramification point at $z = 1$, order 2

1 at ∞ order

$$2g_x - 2 = -8 + \underbrace{2 \cdot 1 + 1 \cdot 2 + 3}_{\text{supposed to be 6}}$$

$$\Rightarrow g_x = 0$$

$$\hat{\Delta} \quad y^N = x^A (1-x)^B (1-xt)^C$$

13.04.2011

$$f(z, w) = w^4 - 2w^2 + 1 - z$$

going around 0

$$f_1(x) \rightarrow f_2(x)$$

going around 1

$1+x$ not changed, so f_1, f_3 are transformed to themselves

$$\text{but } f_2 \leftrightarrow f_4$$

$\pi^{-1}(1) = 3$ points: one is a branch point ($E_2 \cap E_1$) and 2 points on E_1 and E_3 with no branching

at ∞ single branch point of order 3.

$$\Rightarrow g=0 \quad (2g-2 = -8+6)$$

analytically branch points are calculated

$$\begin{cases} f(z_0, w_0) = 0 \\ \frac{\partial f}{\partial w}(z_0, w_0) = 0 \end{cases}$$

$$(z_0, w_0) = \{(0, \pm 1), (1, 0)\}$$

$$f(z, w) = z + 4(w-1)^2 + o(w-1)^3$$

similar +

$$z \sim (w-1)^2$$

$$f(z, w) = (z-1) + 2w^2 + o(w^3)$$

$$f(z, w) = -\frac{1}{1/z} + \frac{1}{(1/w)^4}$$

$$(2-1) + (2-1) + (2-1) + (4-1) = 6$$

$$2g-2 = -2 \cdot 4 + 6$$

$$\Rightarrow g=0$$

Task: Try to determine the genus

$$y^N = x^A (1-x)^B (1-xt)^C$$

$$f(x, y) = y^N - x^A (1-x)^B (1-xt)^C$$

$$\deg_y f(x, y) = N$$

$x = 0, 1, \frac{1}{t}, \infty$ - ramification points

$$y^N \sim x^{A+B+C} \quad \text{at } \infty$$

$$\text{at } 0: y^N \sim x^A$$

$$A = A_0 \Delta_0$$

$$N = N_0 \Delta_0$$

$$(y^{N_0})^{\Delta_0} \sim (x^{\Delta_0})^{A_0}$$

$$\text{order } \Delta_0 (N_0 - 1) = \frac{N}{N_0} (N_0 - 1) = N - \frac{N}{N_0}$$

at 1 and at $\frac{1}{t}$: by analogy we have

$$N(1 - \frac{1}{N_1}) \quad \text{and} \quad N(1 - \frac{1}{N_t}) \quad \text{respectively}$$

$$\text{at } \infty: N(1 - \frac{1}{N_\infty})$$

$$2g-2 = -2N + 4N - N(\frac{1}{N_0} + \frac{1}{N_1} + \frac{1}{N_t} + \frac{1}{N_\infty})$$

Euler characteristics

D^k k -dimensional cell $\xrightarrow{\text{diffeo}}$ $A_j^k \in M$

D^0 - point

D^1 - interval $\longrightarrow$

D^2 - square $\longrightarrow$

D^3 - ...

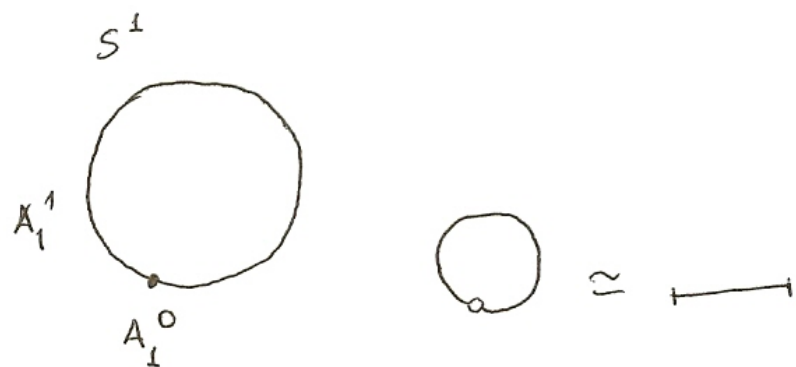
we want to decompose our compact mfd M into a union of A_j^k

$$\chi(M) = \sum_{k=0}^n (-1)^k \# \{A_j^k\}$$

Euler ch.

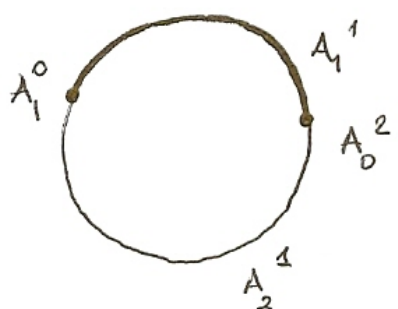
number of cells

doesn't depend on triangulation



$$\chi(S^1) = (-1)^0 \cdot 1 + (-1)^1 \cdot 1 = 1 - 1 = 0$$

in $\mathbb{R}^3$: faces vertices
 $\chi(M) = F + S - A$

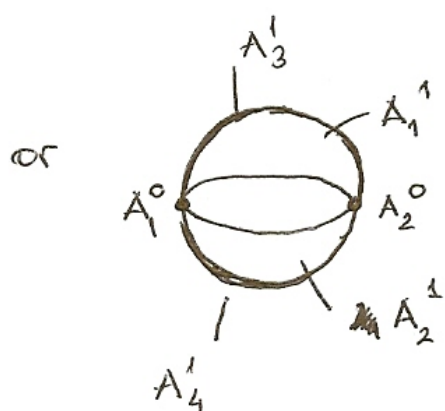


$$\chi(S^1) = (-1)^0 \cdot 2 + (-1)^1 \cdot 2 = 0$$

the genus: $g = 2 - \chi(M)$ $\chi = 2 - 2g$

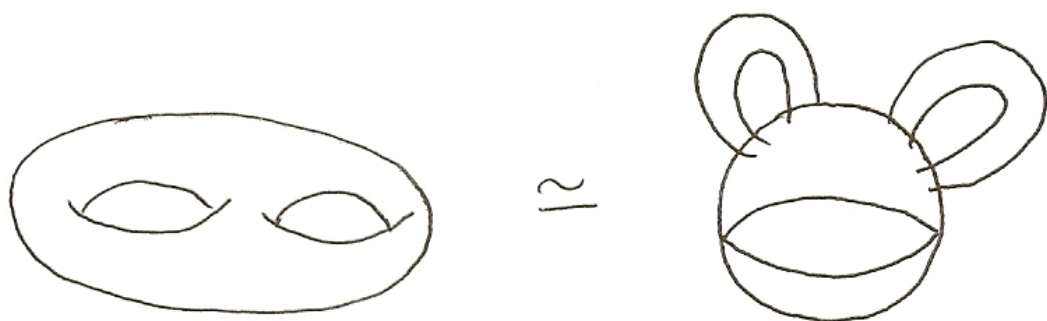
$$\chi(M \times N) = \chi(M) \cdot \chi(N)$$

$\chi(S^2)$

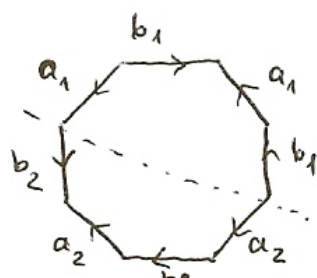
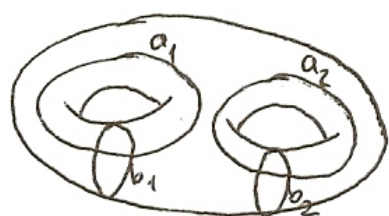
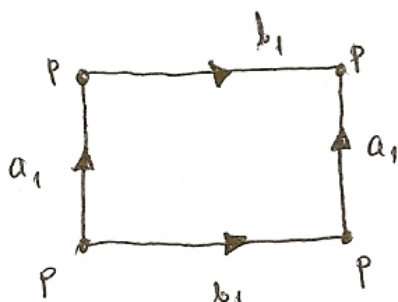
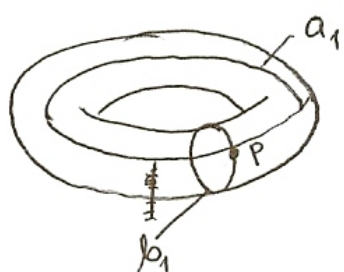


$$\chi(S^2) = 2 - 4 + 4 = 2$$

$$\chi(S^2) = 1 - 1 + 2 = 2$$



$g = 2$
 $\chi(?) = ?$



$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$k \in (0,1)$$

$$K(k) = \frac{\pi}{2} {}_2F_1 \left(\begin{matrix} 1/2 & 1/2 \\ 1 \end{matrix} \middle| k^2 \right)$$

$${}_2F_1 \left(\begin{matrix} a & b \\ c \end{matrix} \middle| x \right) \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} x^j \quad \text{hypergeometric series}$$

Pochhammer symbol

convergent for $|x| < 1$

$$(a)_j = a(a+1)\dots(a+j-1), \quad (a)_0 = 1$$

$${}_mF_n \left(\begin{matrix} a_1 \dots a_m \\ c_1 \dots c_n \end{matrix} \middle| x \right) = \sum \frac{(a_1)_j \dots (a_m)_j}{(c_1)_j \dots (c_n)_j} x^j$$

$$2) \quad {}_2F_1 \left(\begin{matrix} a & b \\ c \end{matrix} \middle| x \right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-b)} \int_0^1 \frac{t^{b-1} (1-t)^{c-b-1}}{(1-tx)^a} dt$$

$$3) \quad x(1-x)y'' + (c - (a+b-1)x)y' - aby = 0$$

$y = y(x)$

$$K(k) \text{ is a solution } (k^3 - k)y''(k) + (3k^2 - 1)y'(k) + ky(k) = 0$$

Conformal transformations

$$u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

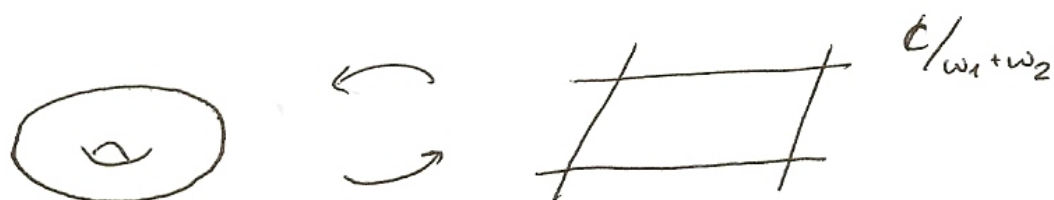
$$K = \int_0^1$$

if one takes a Riemann's surface of $\Delta(t,k) = \sqrt{(1-t^2)(1-k^2t^2)}$



$\Rightarrow$ function sn conformally maps this to the parallelograms of periods

$$\text{sn}(u + 4mk + 2\pi i k') = \text{sn}(u)$$



What happens to Weierstrass function?

$$p(u): (p')^2 = 4p^3 - g_2p - g_3, \quad g_2, g_3$$

$$= 4(p-e_1)(p-e_2)(p-e_3) \quad e_1 + e_2 + e_3 = 0$$

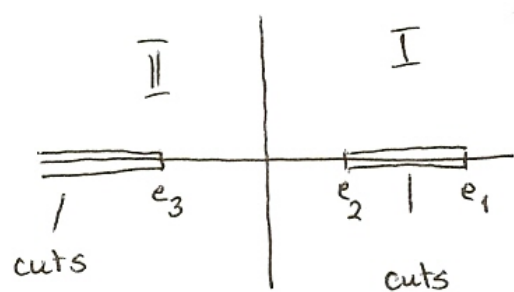
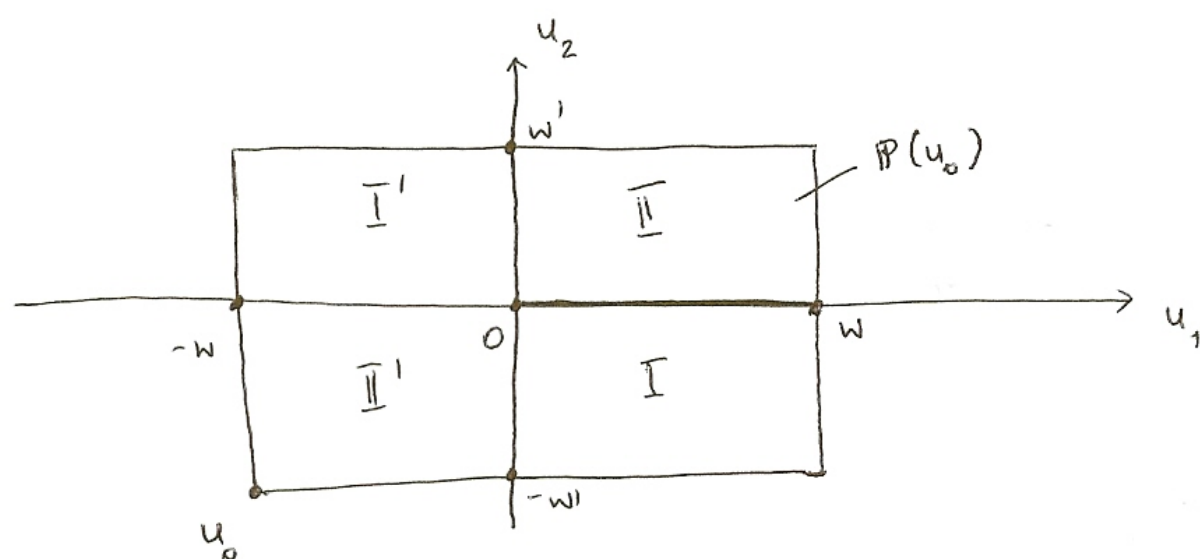
$$\int \frac{dx}{\sqrt{4x^3 - g_2x - g_3}} : p(u) \text{ - inverse to this integral}$$

$$\text{Let } e_1, e_2, e_3 \in \mathbb{R} \quad e_1 > e_2 > e_3$$

$$w = \int_{e_1}^{\infty} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}$$

$2w, 2w'$ - periods of $p(u)$
 $\uparrow$ $\uparrow$
 purely real purely imaginary

$$w' = i \int_{-\infty}^{e_3} \frac{dx}{\sqrt{\dots}}$$



If u increases from 0 to $w \Rightarrow$

z decreases from ∞ to e_1

If u from w to $w + w'$

How to transform conformally a rectangle to a circle of radius 1:

$$z = f(u) = - \frac{\sqrt{e_2 - e_3} + i\sqrt{e_1 - e_2}}{\sqrt{e_2 - e_3} - i\sqrt{e_1 - e_2}} \cdot \frac{p(u) - e_2 + i\sqrt{(e_1 - e_2)(e_2 - e_3)}}{p(u) - e_3 - i\sqrt{\dots}}$$

$$u = tw + t'w'$$

$$0 \leq t \leq 1$$

$$0 \leq t' \leq 1$$

$$e_1 > e_2 > e_3$$

THETA FUNCTIONS (INTRO)

Elliptic functions can be represented by using series

I) Fourier series for entire periodic functions

$f(u)$ - entire 1-periodic function with period $w \neq 0$

$\psi(\xi) \stackrel{\text{def}}{=} \varphi\left(\frac{w}{2\pi i} \ln \xi\right)$ - regular for any $\xi \neq 0, \infty$ (because of $\ln$)

single-valued (because $\ln \rightarrow \ln + 2\pi i$ and $\varphi(u+w) = \varphi(u)$)

$\psi(\xi)$ - single valued, regular in $0 < |\xi| < \infty$

$\psi(\xi) = \sum_{n=-\infty}^{\infty} c_n \xi^n$ convergent in $\rho < |\xi| < R, \rho > 0, R < \infty$

Laurent series

$$\frac{w \ln \xi}{2\pi i} \rightarrow w \frac{\ln \xi}{2\pi i} + \frac{2\pi i}{2\pi i} w$$

going around zero u $u+w$

II Definitions:

$f(u)$ w_1, w_2 - periods

$$w := \frac{1}{2}w_1, \quad w' := \frac{1}{2}w_2, \quad \tau = \frac{w'}{w} = \frac{w_2}{w_1}$$

$$h := \exp(\tau \cdot i\pi) \quad |h| = |e^{\pi i \tau}| = e^{-\pi \operatorname{Im} \tau} < 1 \quad (\Leftrightarrow) \quad \operatorname{Im} \tau > 0$$

$$\xi'(u) = -p(u)$$

$$\xi(u+w_k) - \xi(u) = \eta_k - \text{const.}$$

$$\eta_k = 2\xi\left(\frac{w_k}{2}\right)$$

$$\boxed{\eta_1 w_2 - \eta_2 w_1 = 2\pi i} \quad - \text{Legendre's relation}$$

$$\eta = \frac{1}{2}\eta_1, \quad \eta' = \frac{1}{2}\eta_2$$

$$\eta w' - \eta' w = \frac{\pi i}{2} \quad (\text{in new variables})$$

$$v = \frac{u}{2w}$$

$$z = \exp(\pi i w) = \exp\left(\frac{\pi i u}{2w}\right)$$

$$h^S = e^{\pi i \tau g}$$

$$z^S = \exp(\pi i v g)$$

$$R_1 = \{1, 3, 5, \dots\} \quad - \text{odd integers}$$

$$R_2 = \{2, 4, 6, \dots\} \quad - \text{even}$$

$$\sum_{n \in R_1} c_n = \sum_{k=0}^{\infty} c_{2k+1}, \quad \sum_{n \in R_2} c_n = \sum_{k=1}^{\infty} c_{2k}$$

By To:

$$\xi(u) = \frac{d}{du} \ln \sigma(u)$$

$$\sigma(u) = u \prod_{\tilde{w}} \left(1 - \frac{u}{\tilde{w}}\right) \exp\left(\frac{u}{\tilde{w}} + \frac{u^2}{2\tilde{w}^2}\right) \quad \sigma - \text{entire function with zeros congruent to } 0.$$

$$\sigma(u+2w) = -e^{2\eta(u+w)} \sigma(u) \quad (1)$$

$$\sigma(u+2w') = -e^{2\eta'(u+w')} \sigma(u) \quad (2)$$

in new notation

$$\varphi(u) = e^{au^2+bu} \sigma(u) \quad \text{define } a, b \text{ st. } \varphi(u+2w) = \varphi(u)$$

$$\frac{\varphi(u+2w)}{\varphi(u)} = -\exp\{2(2aw+\eta)(u+w) + 2bw\} \quad b = \frac{\pi i}{2w}$$

$$\frac{\varphi(u+2w')}{\varphi(u)} = -\exp\{2(2aw'+\eta')(u+w') + 2bw'\} \quad a = \frac{-\eta}{2w}$$

$$\frac{\varphi(u+2w')}{\varphi(u)} = -\exp\left\{-\frac{\pi i}{w}(u+w') + \pi i \frac{w'}{w}\right\} = -\exp\left\{-\frac{\pi i u}{w}\right\} = -\frac{1}{z^2}$$

$$z = \exp\left(\frac{\pi i u}{2w}\right)$$

The function $\varphi(u) \sigma(u) \exp\left\{\frac{\eta u^2}{2w} + \frac{\pi i u}{2w}\right\} = z \sigma(u) e^{-\eta u^2/2w}$ has period $2w$

$$\varphi(u+2w) = \varphi(u) \quad \varphi - \text{entire, } 1 - \text{periodic}$$

$$\varphi(u+2w') = -\frac{1}{z^2} \varphi(u)$$

$$\Rightarrow \varphi(u) = \sum_{n=-\infty}^{\infty} A_n \exp\left\{n \frac{2\pi i u}{2w}\right\} = \sum_{n=-\infty}^{\infty} A_n z^{2n}, \quad z = e^{\frac{\pi i u}{2w}}$$

$$\sigma(u) = \frac{1}{\pi} \varphi(u) e^{2u^2/2w} = c \theta_1(v) e^{2u^2/2w}$$

$$\theta_1(v) = i \sum_{n=-\infty}^{\infty} (-1)^n h \left(\frac{2n-1}{2} \right)^2 x^{2n-1}$$

$\sigma \rightsquigarrow$ modify $\varphi \rightsquigarrow \theta_1$

Weierstrass also introduced $\sigma_1(u), \sigma_2, \sigma_3$

$$\sigma_1(u) = e^{2u} \frac{\sigma(w-u)}{\sigma(w)}$$

$$\sigma_2(u) = e^{(2+2')u} \frac{\sigma(w+w'-u)}{\sigma(w+w')}$$

$$\sigma_3(u) = e^{2'u} \frac{\sigma(w'-u)}{\sigma(w')}$$

$$p(u) = -\xi'(u) = \frac{\sigma'^2(u) - \sigma''(u)\sigma(u)}{\sigma^2(u)}$$

- this we already know

$$p(u) - e_1 = \left\{ \frac{\sigma_1(u)}{\sigma(u)} \right\}^2$$

$$p(u) - e_2 = \left\{ \frac{\sigma_2(u)}{\sigma(u)} \right\}^2$$

$$p'^2 = 4\Gamma(p - e_i)$$

$$p(u) - e_3 = \left\{ \frac{\sigma_3(u)}{\sigma(u)} \right\}^2$$

$p(u) - e_k$ have only double zeros and double poles

$\sigma_1, \sigma_2, \sigma_3$ - entire, even

$$\sigma_1(0) = \sigma_2(0) = \sigma_3(0) = 1$$

$$\sigma(0) = 0, \quad \sigma'(0) = 1$$

$$(p')^2 = 4 \frac{\sigma_1^2(u) \sigma_2^2(u) \sigma_3^2(u)}{\sigma^6(u)}$$

$$p' = - \frac{2\sigma_1(u) \sigma_2(u) \sigma_3(u)}{\sigma^3(u)}$$

$$\sigma(u) = c e^{2\frac{u^2}{2w}} \theta_1\left(\frac{u}{2w}\right)$$

$$\sigma_1(u) = c_1 \theta_2(v) \exp\left\{\frac{\eta u^2}{2w}\right\}$$

$$\sigma_2(u) = c_2 \theta_3(v) \exp\left(\frac{\eta u^2}{2w}\right)$$

$$\sigma_3(u) = c_3 \theta_0(v) \exp\left(\frac{\eta u^2}{2w}\right)$$

alternatively

$$\sigma_1(u) = \frac{\theta_2(v)}{\theta_2'(0)} \exp\left(\frac{\eta u^2}{2w}\right)$$

$$\sigma_2(u) = \frac{\theta_3(v)}{\theta_3'(0)} \exp\left(\frac{\eta u^2}{2w}\right)$$

$$\sigma_3(u) = \frac{\theta_0(v)}{\theta_0'(0)} \exp\left(\frac{\eta u^2}{2w}\right)$$

$$\sigma = \frac{2w}{\theta_1'(0)} \theta_1(v) \exp\left(\frac{\eta u^2}{2w}\right)$$

- prim :-)

$$\frac{\sigma_k(u)}{\sigma(u)} = \frac{1}{2w} \frac{\theta_1'(0)}{\theta_{k+1}'(0)} \frac{\theta_{k+1}(v)}{\theta_1(v)} \quad \begin{matrix} k=1,2,3 \\ \theta_4 = \theta_0 \end{matrix}$$

$$\sqrt{p(u) - e_k}$$

Hermite generalized theta function

$$\Theta_{\mu, \nu}(\nu, \tau) = \sum_{n=-\infty}^{\infty} \exp(2\pi i \nu (n + \frac{\mu}{2}) + \pi i \tau (n + \frac{\mu}{2})^2 + \pi i \tau \nu)$$

μ, ν - any complex numbers

$$\text{Im } \tau > 0$$

convergent series $\mathcal{D} \subset \mathbb{C}$

$\Theta_{\mu, \nu}$ - entire function of ν , μ, ν regular for $\text{Im } \tau > 0$

$$\Theta_{\mu, \nu}(\nu, \tau) = \sum_{n=-\infty}^{\infty} e^{\pi i \nu n} h^{\frac{(2n+\mu)^2}{2}} z^{2n+\mu}$$

$$\Theta_0(\nu) = \Theta_4(\nu) = \Theta_{0,1}(\nu, \tau)$$

$$\Theta_1(\nu) = -i \Theta_{1,1}(\nu, \tau)$$

$$\Theta_2(\nu) = \Theta_{1,0}(\nu, \tau)$$

$$\Theta_3(\nu) = \Theta_{0,0}(\nu, \tau)$$

$$\Theta_{\mu, \mu', \nu, \nu'}(\nu, \tau) = \Theta_{\mu, \nu}(\nu + \frac{\nu' + \mu' \tau}{2}, \tau) \cdot \exp\left\{ \pi i \mu' \nu + \pi i \frac{\mu'^2}{4} - \pi i \frac{\mu \nu'}{2} \right\}$$

$$\frac{\partial^2}{\partial \nu^2} \Theta_{\mu, \nu} = 4\pi i \frac{\partial}{\partial \tau} \Theta_{\mu, \nu} \quad (\text{heat equation})$$

$$\Theta_{\mu, \nu+2}(\nu, \tau) = \Theta_{\mu, \nu}(\nu, \tau) \quad ; \quad \Theta_{\mu+2, \nu}(\nu, \tau) = e^{-\pi i \nu} \Theta_{\mu, \nu}(\nu, \tau)$$

any $\Theta_{\mu, \nu}$ can be expressed in terms of Θ_{00} also $\Theta_1, \Theta_2, \Theta_3, \Theta_0$ are dependent

$$m = h^{-1/4} z^{-1} = \exp\left\{ \frac{\pi i \tau}{4} - \pi i \nu \right\}$$

$$k = h^{-1} z^{-2} = \exp\left\{ -\pi i \tau - 2\pi i \nu \right\}$$

	$\nu + \frac{1}{2}$	$\nu + \frac{\tau}{2}$	$\nu + \frac{1}{2} + \frac{\tau}{2}$	$\nu + 1$	$\nu + \tau$	$\nu + 1 + \tau$
Θ_1	Θ_2	$im \Theta_0$	$m \Theta_3$	$-\Theta_1$	$-k \Theta_1$	$k \Theta_1$
Θ_2	$-\Theta_1$	$im \Theta_3$	$-im \Theta_0$	$-\Theta_2$	$k \Theta_2$	$-k \Theta_2$
Θ_3	Θ_0	$m \Theta_2$	$im \Theta_1$	Θ_3	$k \Theta_3$	$k \Theta_3$
Θ_0	Θ_3	$im \Theta_1$	$m \Theta_2$	Θ_0	$-k \Theta_0$	$-k \Theta_0$

$$\Theta_1(\nu + \frac{1}{2} + \frac{\tau}{2}) = m \Theta_3 = \Theta_3(\nu) \cdot e^{-\frac{\pi i \tau}{4} - \pi i \nu}$$

$$\Theta_1(\nu) = 0 \Leftrightarrow \nu \text{ is a zero of } \sigma \quad m_1 w_1 + m_2 w_2 = n + n' \tau$$

From the table we can get zeros of other weierstrass theta's:

$$\Theta_2(\nu) = 0 \quad \nu = n + n' \tau + \frac{1}{2} \quad n, n' \in \mathbb{Z}$$

$$\Theta_2(\nu + \frac{1}{2}) = -\Theta_1$$

$$\Theta_3(\nu) = 0 \Leftrightarrow \nu = n + n' \tau + \frac{1}{2} + \frac{\tau}{2} \quad \text{because } \Theta_3(\nu + \frac{1}{2} + \frac{\tau}{2}) = im \Theta_1$$

$$\Theta_0(v) = 0 \Leftrightarrow v = n + n'\tau + \frac{\tau}{2}, \text{ because } \Theta_0(v + \frac{\tau}{2}) = i\infty \Theta_1$$

$$\sqrt{e_1 - e_k} = \frac{1}{2w} \frac{\Theta_1'(0)}{\Theta_{k+1}(0)} \frac{\Theta_{k+1}(\frac{1}{2})}{\Theta_1(\frac{1}{2})} = (*) \sqrt{p(u) - e_k} = f(a_k)$$

$$u = w_1 w + w'$$

$$\sqrt{e_2 - e_k} = \frac{1}{2w} \frac{\Theta_1'(0)}{\Theta_{k+1}(0)} \frac{\Theta_{k+1}(\frac{1}{2} + \frac{\tau}{2})}{\Theta_1(\frac{1}{2} + \frac{\tau}{2})} \quad k = 1, 2, 3$$

$$(*) = \frac{1}{2w} \frac{\Theta_1'(0) \Theta_0(0)}{\Theta_3(0) \Theta_2(0)}$$

$$\sqrt{e_1 - e_3} = \frac{1}{2w} \frac{\Theta_1'(0) \Theta_3(0)}{\Theta_0(0) \Theta_2(0)}$$

$$\sqrt{e_2 - e_3} = \frac{1}{2w} \frac{\Theta_1'(0) \Theta_2(0)}{\Theta_0(0) \Theta_3(0)}$$

one of the values $\sqrt{e_3 - e_k}$ is expressed in terms of single-valued function of variable $\tau = \frac{w'}{w} = \frac{w_2}{w_1}$, regular in $\text{Im } \tau > 0$

20.04.2011

$$\Theta_{1,2,3,4} \quad \tau' = \frac{w'}{w} = \frac{w_2}{w_1}$$

$$p(w) \quad e_i \quad e_i = \frac{\pi}{w^2} \left(\frac{1}{3} \frac{d}{d\tau} \ln \Theta_1'(0) - \frac{d}{d\tau} \ln \Theta_2(0) \right)$$

$$e_k = \frac{1}{kw^2} \left(\frac{1}{3} \frac{\Theta_1''(0)}{\Theta_1'(0)} - \frac{\Theta_{k+1}'(0)}{\Theta_{k+1}(0)} \right) ; \quad e_1 + e_2 + e_3 = 0$$

Expansions of theta function in products

Θ_3 - regular in $\mathbb{C}$ (except for $z=0, z=\infty$)

$$\Theta_3(v + 1) = \Theta_3(v) \quad \Theta_3(v + i) = h^{-1} z^{-1} \Theta_3(v)$$

$$\Theta_3(v) = \text{const} \prod_{n=1}^{\infty} (1 + h^{2n-1} z^2) (1 + h^{2n-1} z^{-2})$$

all other theta's can be written in infinite products.

Remark: $\wp$ -f, ξ -f, $\wp_2, \wp_3, \wp_n, K, K'$ can be written in Θ -functions.

terms of

for instance $\wp(u, \tau) = \frac{\Theta_2(0)}{\Theta_2(0)} \cdot \frac{\Theta_1(v)}{\Theta_0(v)}$

Jacobi's theta functions are elliptic analogues of exponential functions, doubly quasiperiodic they have representation in terms of series, infinite products.

Elliptic curves:

Plan: addition on a cubic, normal forms

$$y^2 = x(x-1)(x-2)$$

Cubic curve is a particular case of a plane algebraic curve

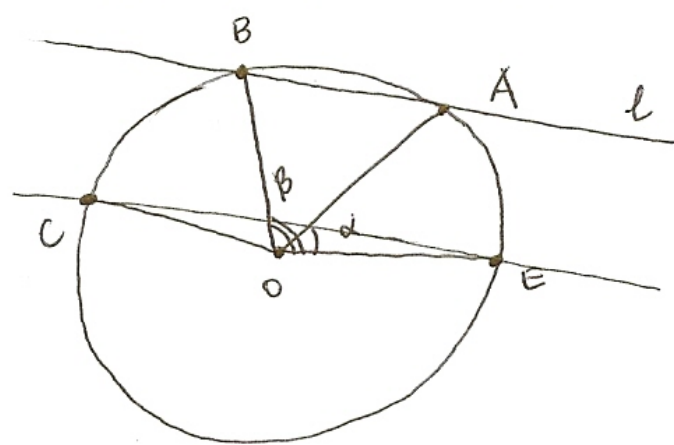
$$\{(x, y) \in \mathbb{R}^2 : f(x, y) = 0, f \in \mathbb{C}[x, y]\}$$

for some plane algebraic curves one has group laws

Example: line l , fix point $O \in l$
 given 2 points $x, y \in l$ we can define z s.t. $Oz = Ox + Oy$
 this operation is a commutative group with respect to addition

circle S : $x: (r \cos \alpha, r \sin \alpha) \in S(O, r)$
 $y: (r \cos \beta, r \sin \beta) \in S$

$$\Rightarrow z: (r \cos(\alpha + \beta), r \sin(\alpha + \beta))$$



$$\angle EOC = \alpha + \beta$$

$E = (r, 0)$ - zero element of group

Analogously one can define addition for any conic (curve of degree 2.)

Example:

1) $y = x^2$ $E = (0, 0)$

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2 + 2x_1x_2)$$

2) $y^2 - x^2 = 1$ $E = (1, 0)$

$$(x_1, y_1) + (x_2, y_2) = (x_1x_2 + y_1y_2, y_1x_2 + y_2x_1)$$

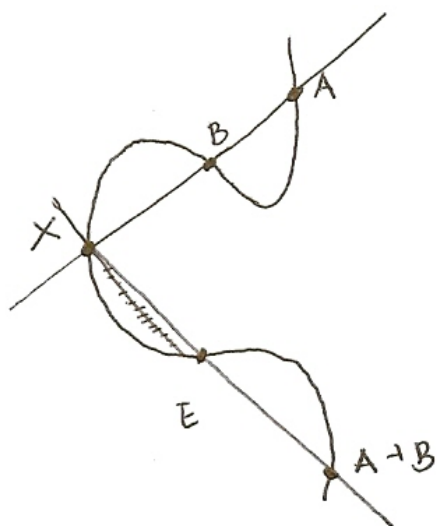
addition for
 $x = \cosh t, y = \sinh t$

A cubic curve is a plane algebraic curve

$$\sum_{i,j} a_{ij} x^i y^j = 0, \quad \max(i+j) = 3$$

< singular (defined later)
 nonsingular

for real x, y if the curve is nonsingular



Fix E , take points A, B to define $A + B$
 one draws a line through A and B ,
 one gets X point and then one
 draws the line through X and E .

Remark: $ax + by + c = 0$ - line (1)

cubic $f(x, y) = 0$ (2)

substituting x (or y from (1) to (2))

we have a third degree equation $\Rightarrow$ 3 roots

Remark: for singular curves one may not define addition for all points.

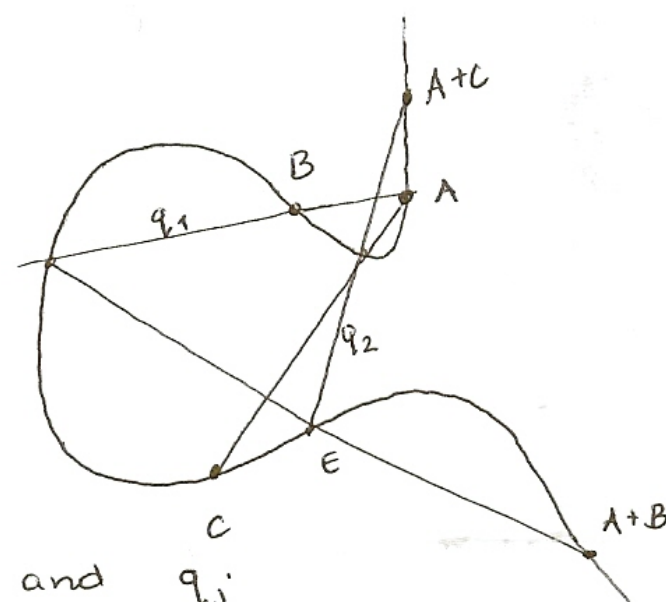
$$A + B = B + A \quad (\text{evident})$$

$$(A + B) + C = (A + C) + B \quad (\text{a picture can be drawn here})$$

$(A+B)+C = (A+C)+B \iff$ intersection points of lines which connect $A+B$ and C and also $A+C$ and B lie on a cubic curve.

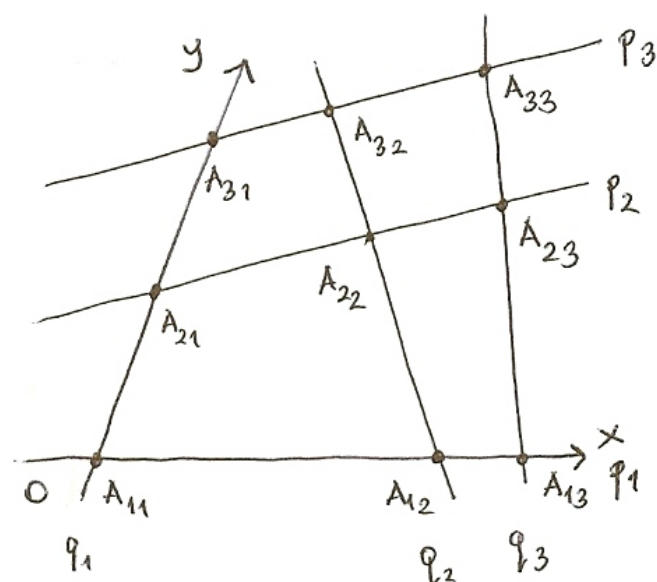
Let line $P_1 = AC$
 $P_2 = E(A+B)$
 $P_3 = B(A+C)$

$q_1 = AB, q_2 = E(A+C), q_3 = C(A+B)$



Let the points A_{ij} of intersection of lines P_i and q_j are pairwise different $i, j \in \{1, 2, 3\}$

Let all A_{ij} except A_{33} lie on a cubic curve $\Rightarrow$ then A_{33} also lies on a cubic.



$P_i(x, y) = 0$
 $q_j(x, y) = 0$ } equation of lines q_j, P_i

$P_1 P_2 P_3 = 0$
 $q_1 q_2 q_3 = 0$ } 3rd degree algebraic plane curve

$\alpha P_1 P_2 P_3 + \beta q_1 q_2 q_3 = 0$ (*)
 cubic passes through all A_{ij} .

Next we show that one can represent any cubic $\underbrace{p(x, y) = 0}_{\text{curve}}$ passing through 8 points A_{ij} in the form (*)

$P_1(x, y) = y$ by affine change of variables
 $q_1(x, y) = x$

functions $P_1(0, y), P_2(0, y), P_3(0, y)$ are zeros at A_{11}, A_{21}, A_{31} on any axis
 $P_1(0, y) = \alpha y P_2(0, y) P_3(0, y)$.

By analogy $P(x, 0) = \beta x q_2(x, 0) q_3(x, 0)$

Consider $Q(x, y) = P(x, y) - \alpha y P_2(x, y) P_3(x, y) - \beta x q_2(x, y) q_3(x, y)$

$Q(0, y) = P(0, y) - \alpha y P_2(0, y) P_3(0, y) \equiv 0$

$Q(x, y) = a_0(y) + a_1(y)x + a_2(y)x^2 + a_3(y)x^3$

for $\forall y, Q(0, y) = 0$
 $\Rightarrow a_0(y) = 0$

$\Rightarrow Q(x, y)$ is divisible by x

similarly $Q(x, y)$ is divisible by y .

$\Rightarrow Q(x, y) = xy Q_1(x, y)$

linear in x, y or constant

At A_{22}, A_{23}, A_{32}
 $xy \neq 0$

$P(x, y) = 0$

$P_2 P_3, q_2 q_3 \Rightarrow Q = 0$

$\Rightarrow Q_1(x, y) = 0$ at those points

points A_{22}, A_{32}, A_{23} are not collinear

$$\deg Q_1 \leq 1$$

$$Q_1 \equiv 0$$

it means that we may represent

$$P = \alpha p_1 p_2 p_3 + \beta q_1 q_2 q_3$$

$$A_{33} \in P(x, y) = 0.$$

Any cubic curve passing through A_{ij}

$$\alpha p_1 p_2 p_3 + \beta q_1 q_2 q_3 = 0 \quad \text{"pencil of cubic"}$$

if points are not pairwise different one needs to modify a proof taking into account multiple roots of p .

$$\begin{pmatrix} Q_1 = 0 \\ xy \neq 0 \end{pmatrix}$$

Generalization for curves of degree ≥ 3

Theorem: A_{ij} are intersection points of p_i, q_j $i, j \in \{1, \dots, n\}$, $n \geq 3$
pairwise distinct

Let A_{ij} for $i+j < n+3$ lie on a curve of degree n $P_n(x, y) = 0$

$\Rightarrow$ all others belong to the curve.

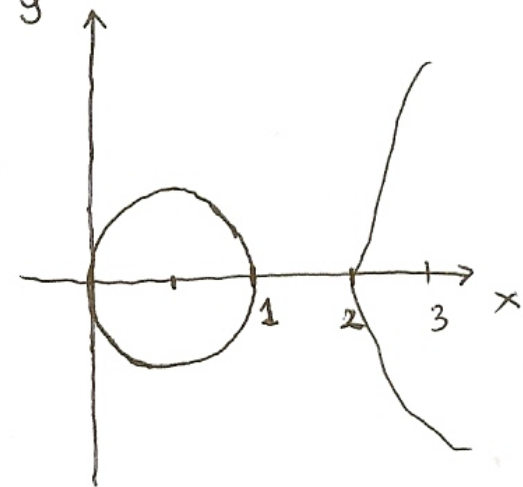
Proof: induction by n , $n=3$ - before

$$P_n(x, y) = \alpha p_1 \dots p_n \quad F$$

$$P_n(x, 0) = \beta q_1 \dots q_n \quad G$$

$$Q_n = P_n - F - G = xy Q_{n-2} \equiv 0$$

$$y^2 = x(x-1)(x-2)$$



$$x = \frac{1}{2} \Rightarrow y^2 = \frac{3}{8}$$

2 integration points

$$\mathbb{RP}^2 \quad (x, y, z) \sim (\lambda x, \lambda y, \lambda z), \quad \lambda \neq 0$$

homogenous coords

$$y^2 z = (x-z)(x-2z)$$

$$z=1 \quad (x, y, z) \sim \left(\frac{x}{z}, \frac{y}{z}, 1\right)$$

$$\sum a_{ij} x^i y^j = 0$$

in $\mathbb{P}^2$

$$\sum a_{ij} x^i y^j z^{n-i-j} = 0$$

$$n = \max(i+j)$$

$$z=0 \quad y = \pm \sqrt{\frac{3}{8}} x$$

3 points of intersection: $\left(\frac{1}{2}, \sqrt{\frac{3}{8}}, 1\right); \left(\frac{1}{2}, -\sqrt{\frac{3}{8}}, 1\right); \underbrace{(0, 1, 0)}_{\text{point at } \infty}$

$$\text{if } y=1 \quad x \sim x^3$$

$\Rightarrow$ any curve in the projective plane $\frac{ax+by+cz=0}{\text{either}}$ belongs to a cubic

curve or intersects it in three (complex) points $\begin{cases} 3 \text{ real roots} \\ 1 \text{ real} \end{cases}$

counting multiplicity

$$\sum_{i+j+k=3} a_{ij} x^i y^j z^k = 0$$

If $c \neq 0$ $z = \alpha x + \beta y$ substituting into P

$$Q(x,y) = \sum b_p x^p y^{3-p} = \begin{cases} b_p = 0 \quad \forall p \Rightarrow Q \text{ is divisible by } ax+by+cz=0, \\ b x^r y^s (x-t_1 y) \dots (x-t_m y) \quad r+s+m=3 \Rightarrow \text{point } (0,1,\beta) \text{ of} \\ \text{multiplicity } r, \text{ point } (0,1,\alpha) \text{ of multiplicity } s, \\ \text{or point } (t_k y, y, \alpha(t_k y) + \beta y) \sim \\ \sim (t_k, 1, \alpha t_k + \beta) \end{cases}$$

$A+A = ?$

the same method but instead of line AB we take tangent line.

Let $P = (p_1, p_2, p_3)$ is a point on $F(x,y,z)=0$, $F(P)=0$

$x = (x_1, x_2, x_3)$

px line $P+tx=0$

$$F(P+tx) = F(P) + at + bt^2 + ct^3 = Q(t)$$

$a = \sum \frac{\partial F}{\partial x_i}(P) x_i = 0$ tangent line

$$b = \frac{1}{2} \sum \frac{\partial^2 F}{\partial x_i \partial x_j}(P) x_i x_j$$

if $\frac{\partial F}{\partial x_i} \neq 0$ for at least one $i \Rightarrow P$ is nonsingular point

$H(x) = \det \left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right) = 0$ - Hessian (cubic curve)

$$f(x,y) = a_0 y^3 + a_1(x) y^2 + a_2(x) y + a_3(x)$$

$$h(x,y) = b_0 y^3 + b_1(x) y^2 + b_2(x) y + b_3(x)$$

a_k, b_k - polynomials of degree at most k , $0 \leq k \leq 3$

to find intersection points of f and h ?

resultant

$$\text{Res}(f,h) = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \end{vmatrix} = p(x)$$

$$p(x) = 0 \Rightarrow x_1, \dots, x_k$$

$$f(x_i, y) = 0$$

Normal forms of nonsingular cubic curve

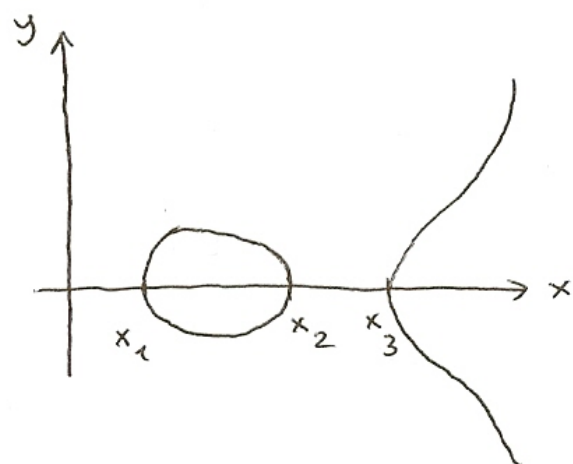
all points are nonsingular

Over $\mathbb{RP}^2$ any nonsingular cubic curve can be reduced (by proj. transformations) to either

$y^2z = x^3 + pxz + qz^3$ (Weierstrass form) $x^3 + px + q$ has no multiple roots
 $x^3 + y^3 + z^3 = 3\lambda xyz$, $\lambda^3 \neq 1$ - otherwise 3 lines

Singular curves

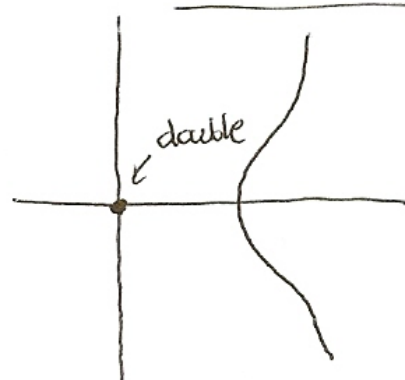
nonsingular $y^2 = (x-x_1)(x-x_2)(x-x_3)$



$$x_i \neq x_j$$

$$x_1 < x_2 < x_3$$

if $x_1 \rightarrow x_2$



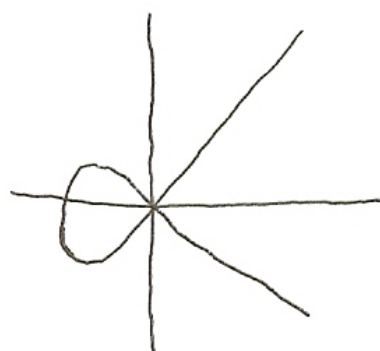
$$\tilde{y}^2 = \tilde{x}^2(\tilde{x}-1)$$

rational par.

here we have elliptic parametrization

$$(p')^2 = 4(p-e_1)(p-e_2)(p-e_3)$$

if $x_2 \rightarrow x_3$

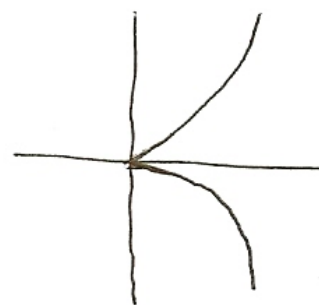


$$\tilde{y}^2 = \tilde{x}^2(\tilde{x}+1)$$

rational par

if $\begin{matrix} x_1 \rightarrow x_3 \\ x_2 \rightarrow x_3 \end{matrix}$

$$\tilde{y}^2 = \tilde{x}^3$$



rational par

in all those cases:

0 - singular point
(no addition for it)

In singular we have rational parametrization

if $y^2 = x^3$ $\begin{cases} x = t^{-2} \\ y = t^{-3} \end{cases}$

if $ax + by + c = 0$
 $c \neq 0$

$$ct^3 + at + b = 0 \Rightarrow t_1, t_2, t_3 \quad t_1 + t_2 + t_3 = 0$$

let's denote $E(\mathbb{P}^1, \infty)$ for $t_E = 0$, this is a singular point

let's take some points A, B corresponding to t_A, t_B

line AB intersects cubic in some point X s.t.

$$t_A + t_B + t_X = 0$$

The line EX intersects the cubic in point $A+B$

$$t_E + t_X + t_{A+B} = 0$$

"

$$0 \quad t_{A+B} = -t_X = t_A + t_B$$

$x^3 = y^3$ we have natural group of addition over $\mathbb{R}$

$$t_{A+B} = t_A + t_B$$

$$y^2 = x^2(x+1)$$

$$y = tx$$

$$t^2 x^2 = x^2(x+1)$$

$$x = t^2 - 1, \quad y = tx = t^3 - t$$

rational

parametrization

$$ax + by + c = 0$$

$$y^2 = x^2(x+1)$$

$$a(t^2 - 1) + b(t^3 - t) + c = 0$$

if $b \neq 0 \Rightarrow t_1, t_2, t_3$ - three roots s.t.

$$t_1 t_2 + t_2 t_3 + t_3 t_1 = -1$$

if one defines $t := \frac{1+\tau}{1-\tau}$

$$\tau_1 \tau_2 \tau_3 = 1$$

$\tau_E = 1 \Rightarrow E$ - infinity point

line AB intersects a cubic in X

$$\tau_A \tau_B \tau_X = 1$$

$$\tau_X \tau_E \tau_{A+B} = 1 \Rightarrow \tau_{A+B} = \tau_A \cdot \tau_B$$

with $x^2(x+1) = y^2$ we have a natural structure of $(\mathbb{R}, \cdot)$

4.08.2011

Def: $u \in \mathbb{N}$ jest liczbą kongruentną, jeśli istnieje trójkąt prostokątny o polu u i bokach wymiernych

$$a, b, c \in \mathbb{Q}, \quad a^2 + b^2 = c^2, \quad ab = 2u$$

$$x = \frac{u(a+c)}{b}, \quad y = \frac{2u^2(a+c)}{b^2} \Rightarrow y^2 = x^3 - u^2 x$$

$$a = \frac{x^2 - u^2}{y}, \quad b = \frac{2ux}{y}, \quad c = \frac{x^2 + u^2}{y}$$

wniosek:

n jest liczbą kongruentną $\Leftrightarrow F(x, y) \in \mathbb{Q}^2 : y \neq 0$

$y^2 = x^3 - n^2 x$ - to jest krzywa eliptyczna

Def: Jeżeli charakterystyka ciała K jest równa 2 i wielomian $x^3 + ax^2 + bx + c$ nie ma pierwiastków podwójnych to krzywą eliptyczną nad K nazywamy zbiór punktów spełniających równanie:

$$y^2 = x^3 + ax^2 + bx + c$$

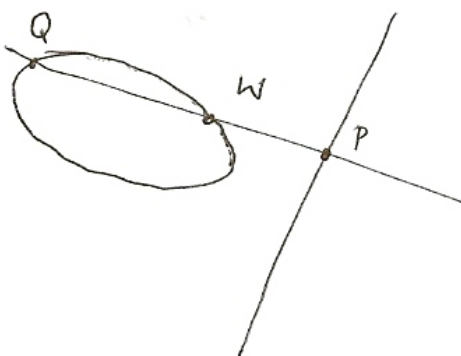
wraz z pewnym punktem wyróżnionym "5" - nazywany punktem w nieskończoności.

2 - stopnia:

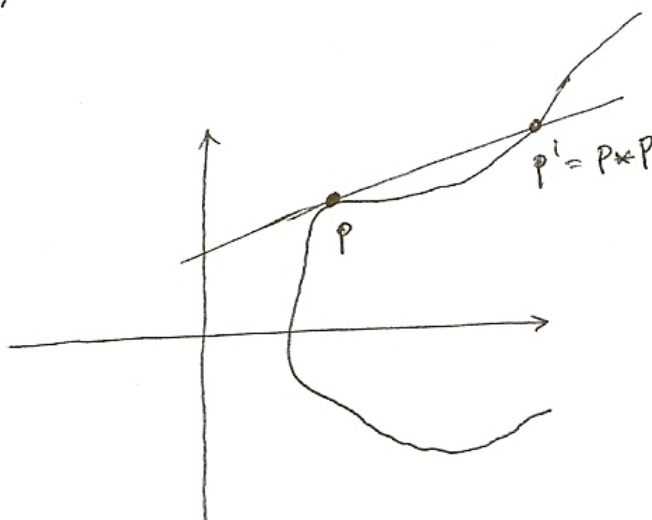
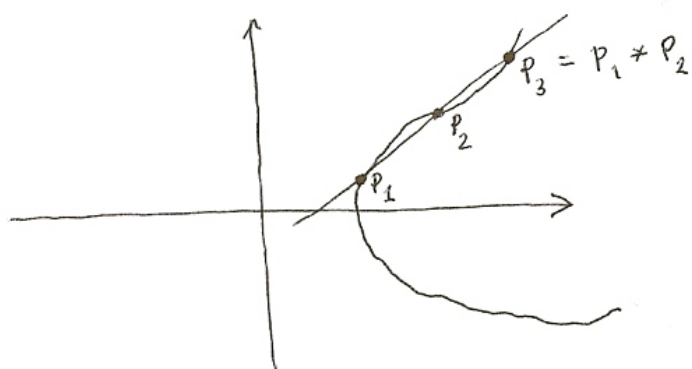
$$S: \sum_{i+j \leq 2} a_{ij} x^i y^j = 0$$

$$L: y + ax + b = 0$$

$$a_{ij}, a, b \in \mathbb{Q}$$

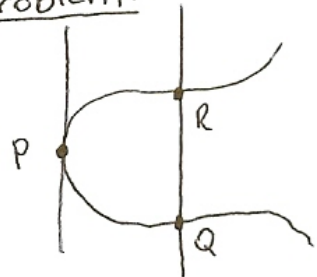


3 - stopnia: ($y^2 = x^3 + ax^2 + bx + c$)



Uwaga: Para $(E, *)$ nie ma struktury grupy (nie istnieje element zerowy)

Problem:



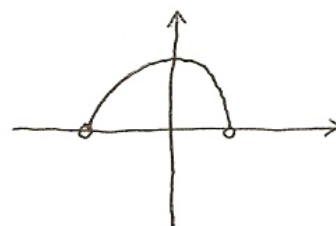
$$P * P = ?$$

$$R * Q = ?$$

Odp: Przestrzeń wektorowa

Def: $x, y \in \mathbb{R}^n$ $x \sim y \iff \exists t \in \mathbb{R} : x = ty$

Def: $\mathbb{P}^1 = \frac{\{(a, b) \in \mathbb{R}^2 : a \neq 0 \vee b \neq 0\}}{\sim}$



Stw: $\mathbb{P}^1 \xleftrightarrow["na"]{1 \neq 1} \{\text{kierunki prostych w } \mathbb{R}^2\}$
 $[a, b] \leftrightarrow ay = bx$

Def: $\mathbb{P}^2 = \frac{\{(a, b, c) \in \mathbb{R}^3 : a \neq 0 \vee b \neq 0 \vee c \neq 0\}}{\sim}$

stw:

$$\mathbb{R}^2 \leftrightarrow \mathbb{R}^2 \cup \mathbb{P}^1$$

$$[a, b, c] \mapsto \begin{cases} (\frac{a}{c}, \frac{b}{c}) \in \mathbb{R}^2 & \text{gdz } c \neq 0 \\ [a, b] \in \mathbb{P}^1 & \text{w.p.p.} \end{cases}$$

$$[x, y, 1] \leftrightarrow (x, y) \in \mathbb{R}^2$$

$$[A, B, 0] \leftrightarrow (A, B) \in \mathbb{P}^1$$

$$E: y^2 = x^3 + ax^2 + bx + c$$

$$y^2z = x^3 + ax^2z + bxz^2 + cz^3$$

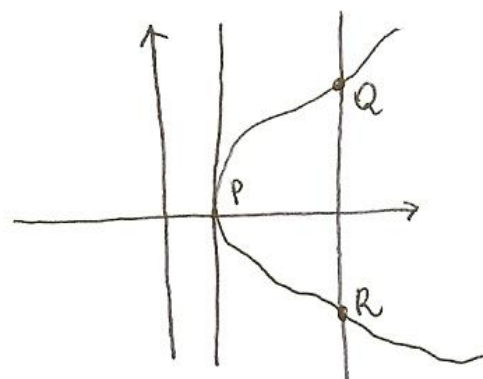
$$\mathbb{P}^1 = \{[x, y, z] : z \neq 0\}$$

$$E \cap \mathbb{P}^1 = \{[x, y, z] : z=0 \wedge x^3=0\} = [0, 1, 0] \mapsto [0, 1] \in \mathbb{P}^1$$

$[0, 1]$ wyznacza kierunek prostej $x=0$

wniosek: $\sigma = [0, 1] \in \mathbb{P}^1$

mała uwaga notacyjna: $\sigma = 0$



$$P * P = \sigma$$

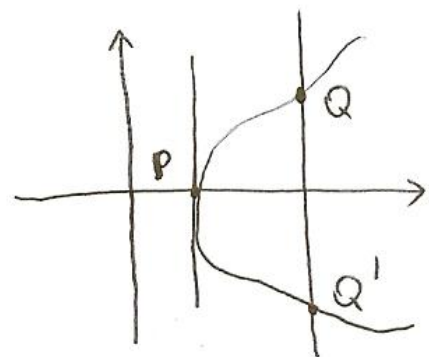
$$Q * R = \sigma$$

$$E: \{(x, y) \in \mathbb{R}^2 : y^2 = x^3 + ax^2 + bx + c\} \cup \{[0, 1]\} \in \mathbb{R}^2 \cup \mathbb{P}^1$$

Def: $P_1, P_2 \in E$, $P_1 + P_2 := (P_1 * P_2) * \sigma$

stw: $(E, +)$ - jest grupą abelową

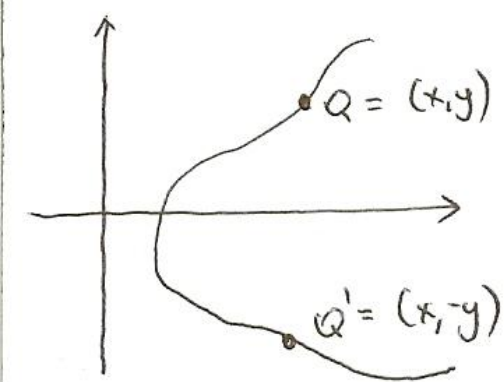
dowód: 1° element neutralny 0



$$Q + 0 = Q$$

$$Q + 0 = ((Q * 0) * \sigma) = Q' * \sigma = Q$$

2° element przeciwny



$$Q + Q' = ((Q * Q') * \sigma) = \sigma * \sigma = 0$$

3° łączność - było pokazane

4° abelowości - oczywiście

Przykład: $y^2 = x^3 + 17$, $P_1 = (-1, 4)$, $P_2 = (2, 5)$

$y = \frac{1}{3}x + \frac{13}{3}$ - prosta łącząca P_1 i P_2

Wstawiając do równania krzywej otrzymujemy

$$\frac{1}{9}x^2 + \frac{26}{9}x + \left(\frac{13}{3}\right)^2 = x^3 + 17$$

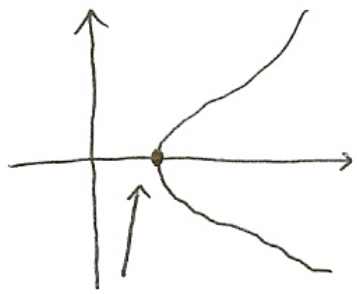
$$-1 + 2 + x_3 = \frac{1}{9}$$

$$(x_3, y_3) = \left(-\frac{8}{9}, \frac{109}{27}\right)$$

$$P_1 + P_2 = \left(-\frac{8}{9}, -\frac{109}{27}\right)$$

Uwaga: $\{0\} \subseteq E(\mathbb{Q}) \subseteq E(\mathbb{R}) \subseteq E(\mathbb{C})$

Def: Niech $P \in E(\mathbb{C})$ mówimy, że P jest ^(ma nęcl) $m \in \mathbb{Z}$, jeśli $m \cdot P = O$



ten element
ma rząd 1.

Def: Wyrażnik wielomianu $f(x) = x^3 + ax + b$ to $-4a^3 - 27b^3 := D$

Twierdzenie (Nagell - Lutz ~ 1935)

$(x, y) \in \mathbb{Q}^2$ jest punktem skończonego rzędu na krzywej eliptycznej o współczynnikach w $\mathbb{Z}$, to $x, y \in \mathbb{Z}$: $y=0$ lub $y \mid D$.

Twierdzenie (Mazur):

Jeśli $E(\mathbb{Q})$ zawiera punkt skończonego rzędu m , to $1 \leq m \leq 10$ lub $m = 12$.

Twierdzenie (1923 - Mordell - "dzieło życia") (sugestia Poincaré 1901)

Niech E będzie krzywą eliptyczną. Wówczas $E(\mathbb{Q})$ jest skończonym generowanym

$$E(\mathbb{Q}) \stackrel{\text{izo}}{\simeq} \mathbb{Z}^r \oplus T_E (\simeq \mathbb{Z}^r \oplus \mathbb{Z}_{p_1^{a_1}} \oplus \dots \oplus \mathbb{Z}_{p_n^{a_n}})$$

Def: r_E - rząd krzywej eliptycznej

T_E - grupa punktów o skończonym rzędzie $E(\mathbb{Q})$ (grupa torsyjna)

wniosek:

$\exists P_1, \dots, P_r \in E(\mathbb{Q})$ t.z. $\forall P \in E(\mathbb{Q}) \exists a_i \in \mathbb{Z}, \tilde{P} \in T_E$

$$P = \sum_{i=1}^r a_i P_i + \tilde{P}$$

Def: wielkość punktu wymiernego. Niech $\frac{m}{n} \in \mathbb{Q}$

$$H\left(\frac{m}{n}\right) = \max\{ |m|, |n| \}$$

$$h\left(\frac{m}{n}\right) = \log H\left(\frac{m}{n}\right)$$

Dowód twierdzenia:

Lemat 1: $\forall M \in \mathbb{R}$ zbiór $\{P \in E(\mathbb{Q}) : h(P) \leq M\}$ jest skończony

Lemat 2: Niech $P_0 \in E(\mathbb{Q})$ wówczas $\exists K_0(P_0, a, b) \in \mathbb{R} : h(P + P_0) \leq 2h(P) + K_0$
 $\forall P \in E(\mathbb{Q})$

Lemat 3: Istnieje $K(a, b, c)$ t.z. $h(2P) \geq 4h(P) - K$, $\forall P \in E(\mathbb{Q})$

Lemat 4: Indeks $(E(\mathbb{Q}) : 2E(\mathbb{Q}))$ jest skończony.

Z lematu 4 dostajemy istnienie $Q_1, \dots, Q_n \in E(\mathbb{Q}) : \forall P \in E(\mathbb{Q})$

$$\exists i \in \{1, \dots, n\} : P - Q_i \in 2E(\mathbb{Q})$$

zatem

$$\begin{aligned} P - Q_{i_1} &= 2P_1 \\ P_1 - Q_{i_2} &= 2P_2 \\ &\vdots \\ P_{m-1} - Q_{i_m} &= 2P_m \end{aligned}$$

$$P = Q_{i_1} + 2Q_{i_2} + 4Q_{i_3} + \dots + 2^{m-1}Q_{i_m} + 2^m P_m$$

Pokażemy, że $h(P_j) \leq h(P_{j-1})$

Lemat 2 $\Rightarrow \forall i \in \{1, \dots, n\}, h(P - Q_i) \leq 2h(P) + k_i, \forall P \in E(\mathbb{Q})$

$$k' = \max \{k_1, \dots, k_n\} \Rightarrow h(P - Q_i) \leq 2h(P) + k'$$

Lemat 3

$$4h(P_j) \leq h(2P_j) + k = h(P_{j-1} - Q_{ij}) + k \stackrel{L2}{\leq} 2h(P_{j-1}) + k' + k$$

$$\Rightarrow 4h(P_j) \leq 2h(P_{j-1}) + k' + k$$

$$h(P_j) \leq \frac{3}{4}h(P_{j-1}) - \frac{1}{4}(h(P_{j-1}) - k' - k)$$

~~$$h(P_j) \leq \frac{3}{4}h(P_{j-1}) - \frac{1}{4}(h(P_{j-1}) - k' - k)$$~~
$$\Rightarrow \exists R \in \mathbb{N} \quad h(P_R) < k' + k$$

$$\Rightarrow P = \sum_{i=1}^n a_i Q_i + 2^R P_R, \text{ gdzie } h(P_R) < k' + k$$

Zatem zbiór $\{Q_1, Q_2, \dots, Q_n\} \cup \{P_R \in E(\mathbb{Q}) : h(P_R) < k' + k\}$

Przykład: $E: y^2 = x^3 + 877x$ rank $(E(\mathbb{Q})) = 1$

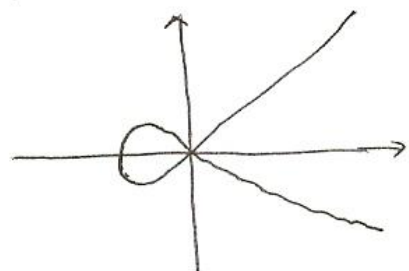
zbiór generatorów $\{(0,0) \cup \left(\frac{61277608387847368101}{7884153586063006240} \right)^2\}$

rekordy: do 1945 $r_E = 3$ Biling

do 2000 $r_E = 24$ Martini McMillen

obecnie nie wiadomo

$$E: y^2 = x^2(x+1)$$



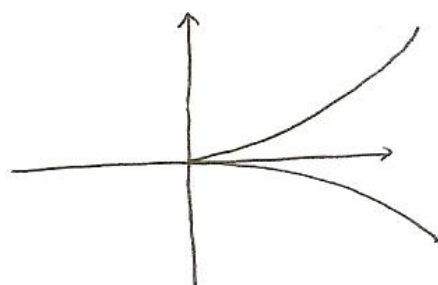
$$x = r^2 - 1$$

$$y = r^3 - r$$

$$\mathbb{Q} \ni r \leftrightarrow (x, y) \in E$$

$$\mathbb{Q}_+ \xleftrightarrow{\text{nomeo}} E / \{P\}$$

$$y^2 = x^3$$



$$x = t^2$$

$$y = t^3$$

$$\mathbb{Q}^* \leftrightarrow E \setminus \{(0,0)\}$$

$$E: y^2 = x^3 - n^2x = x(x+n)(x-n)$$

$$\{O, (0,0), (n,0), (-n,0)\} \in T_E$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2$$

Fakt: $\# T_E = 4$

n jest liczbą kongruentną $\Leftrightarrow r_E > 0$

"struktura $E(\mathbb{Q})$ "

$$\sum_{0 \leq i+j \leq 3} a_{ij} x^i y^j \xrightarrow[\text{kwadratowe}]{\text{przekształcenie}} y^2 = x^3 + nx^2 + bx + c$$

$$E: \sum a_{ij} x^i y^j = 0$$

$$x \rightarrow x - \frac{1}{3}a \rightarrow 4x$$

$$y \rightarrow 4y$$

$$y^2 = x^3 + nx^2 + bx + c \xrightarrow{\quad} y^2 = 4x^3 - a'x - b' : E_W$$

forma normalna
Weierstrassa

$$\exists (x, y) \in E \subseteq \mathbb{C} \times \mathbb{C}$$

$$x, y \in \mathbb{C}$$

uwaga:

stałe a', b' jednoznacznie wyznaczają okresy funkcji weierstrassa ω_1, ω_2 ,
 odwrotnie też prowadzi: $a' = 60 \sum_{\omega \neq 0} \frac{1}{\omega^4}$, $b' = 140 \sum_{\omega \neq 0} \frac{1}{\omega^6}$.

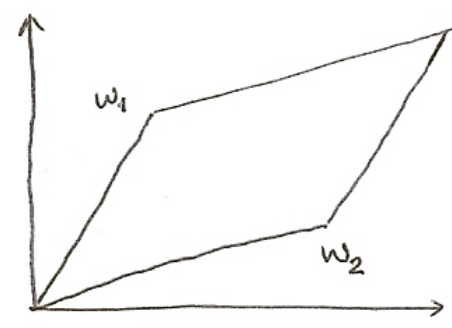
$$P = \{a\omega_1 + b\omega_2 : 0 \leq a, b \leq 1\}$$

$$A: P \rightarrow E_w$$

$$A(u) = (p(u), p'(u)) \text{ gdy } u \neq \omega_1, \omega_2$$

$$A(u) = 0 \text{ w.p.p.}$$

$$P \xrightleftharpoons[A]{\text{bijekcja}} E_w$$



$$E_w: y^2 = 4x^3 - g_2x - g_3$$

$$P_1 = (p(u_1), p'(u_1)) \quad P_2 = (p(u_2), p'(u_2)) \quad , \quad P_1 + P_2 = (x_3, y_3)$$

$$y = \underbrace{\frac{p'(u_2) - p'(u_1)}{p(u_2) - p(u_1)}}_{\lambda} x + c \quad , \quad c = \text{const}$$

$$x_3 + p(u_1) + p(u_2) = \frac{1}{4} \lambda^2$$

$$x_3 = \frac{1}{4} \left(\frac{p'(u_2) - p'(u_1)}{p(u_2) - p(u_1)} \right)^2 - p(u_1) - p(u_2)$$

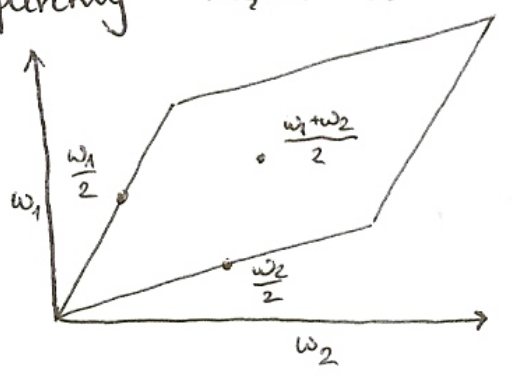
$$x_3 = \cancel{p(u_1)} + \cancel{p(u_2)} + p(u_1 + u_2) \quad (\text{Addition theorem})$$

$$y^3 = p'(u_1 + u_2)$$

wniosek: $(p(u_1 + u_2), p'(u_1 + u_2)) = (p(u_1), p'(u_1)) + (p(u_2), p'(u_2))$

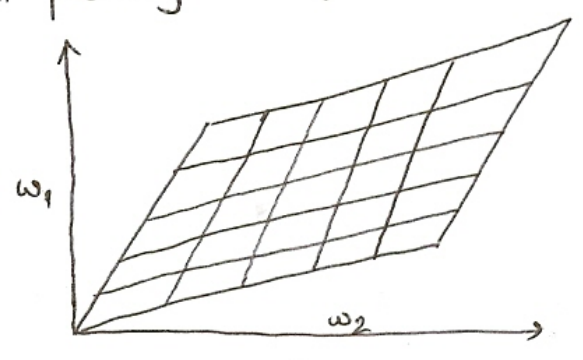
czyli $A(u_1 + u_2) = A(u_1) + A(u_2)$

"punkty rzędu 2"



$$A(2u) = A(u) + A(u)$$

"punkty rzędu 5"



Def: $E(n)$ - punkty rzędu n na E

wniosek: $E(n) \simeq \mathbb{Z}_n \oplus \mathbb{Z}_n$

Podsumowanie: Krzywe eliptyczne znajdują zastosowania w

- 1) Kryptografii
- 2) dowodzie wielkiego Twierdzenia Fermata
- 3) Hipotezie Riemanna
- 4) Algebraicznej teorii liczb
- 5) Geometrii Algebraicznej
- 6) Teorii Liczb

Problemy otwarte:

- 1) Istnienie punktu wymiernego na krzywej eliptycznej
- 2) Podać rząd krzywej eliptycznej

Przydatne programy: Sage, Mapple

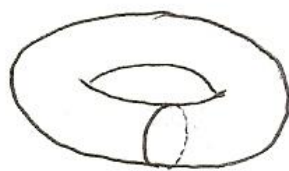
Referat był przygotowany na podstawie:

Joseph H. Silverman, John Tate, "Rational Points on Elliptic Curves".

Hesse normal form:

$$C(x) = \{ [t_0 : t_1 : t_2] \in \mathbb{P}^2 \mid t_0^3 + t_1^3 + t_2^3 - 3xt_0t_1t_2 = 0 \} \quad \text{elliptic curve genus 1}$$

$x \in \mathbb{C} \setminus \{1, \omega, \omega^2\}$
 $\omega = \frac{1}{\sqrt[3]{-1}}$



γ - one of the cycles

$$\varphi = \frac{t_0 dt_1 - t_1 dt_0}{t_2^2 - xt_0t_1} \quad \text{nowhere vanishing holomorphic form}$$

$$p(x) := \int_{\gamma(x)} \varphi \quad \text{period satisfies ODE } H\left(\frac{1}{3}, \frac{1}{3}\right)$$

$$H(\alpha, \beta): (1-x^3) \frac{d^2}{dx^2} h(x) - (3\alpha + 3\beta + 1)x^3 \frac{d}{dx} h(x) - 9\alpha\beta x h(x) = 0$$

Heun's ODE

- 1) Calculate $\frac{d}{dx} p(x)$
- 2) Calculate $\frac{d^2}{dx^2} p(x)$

$$\varphi \text{ can be rewritten as } \frac{d\left(\frac{t_1}{t_0}\right)}{\partial(t_0^3 + t_1^3 + t_2^3 - 3xt_0t_1t_2) / \partial t_2}$$

$$(u, v) = \left(\frac{t_1}{t_0}, \frac{t_2}{t_0}\right) \in \mathbb{R}^2$$

$$q(x, u, v) = 1 + \frac{u^3}{x} + v^3 - 3xuv = 0 \quad v = v(x, y)$$

$$p(x) = \int \frac{du}{v^2 - xu}$$

$$\text{How to differentiate } \frac{d}{dx} \int \psi(x, u, v) du = \int \left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial v} \frac{\partial v}{\partial x} \right) du$$

$$\frac{\partial v}{\partial x} = ? \quad \frac{\partial v}{\partial y} = ?$$

$$q = q(x, u, v) = 0$$

$$dq = q'_u du + q'_v dv = 0$$

$$\Rightarrow dv = -\frac{q'_u}{q'_v} du = -\frac{u^2 - xv}{v^2 - xu} du$$

$$3v^2(x, u) \frac{\partial v(x, y)}{\partial x} - 3uv(x, u) - 3xu \frac{\partial v}{\partial x}(x, u) = 0$$

$$\frac{\partial v(x, y)}{\partial x} (3v^2(x, u) - 3xu) = 3uv(x, u)$$

$$\frac{\partial v(x, y)}{\partial x} = \frac{uv(x, u)}{v^2(x, u) - xu}$$

$$\frac{d}{dx} \int \psi(x, u, v) du = \int \left\{ \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial v} \cdot \frac{\partial v}{\partial x} \right\} du = \int \left\{ \left(\frac{\partial}{\partial x} + \frac{uv}{v^2 - xu} \cdot \frac{\partial}{\partial v} \right) \psi \right\} du$$

$$\frac{d}{dx} \int p(x) = \int \frac{-u(v^2 + xu)}{(v^2 - xu)^3} du$$

$$\frac{d^2}{dx^2} p(x) = \int \frac{2xu^3(5v^3 + xu)}{(v^2 - xu)^5} du$$

Now we'll check if $p(x)$ satisfies

$$(1-x^2) \frac{d^2}{dx^2} p(x) - 3x^2 \frac{d}{dx} p(x) - xp(x) = 0$$

$$\left[(1-x^2) \frac{d^2}{dx^2} - 3x^2 \frac{d}{dx} - x \right] p(x) = x \int \eta(x, u, v) du$$

1) try to reduce it modulo curve $q(x, u, v) = 0$

$$\eta(x, u, v) = \frac{2xu^4 - (9x^3 - 10)u^3v^2 - 9x^2u^2v^4 + 7xuv^6 - v^8}{(v^2 - xu)^5}$$

2) w - n form - closed

$$dw = 0$$

exact

$$\exists (n-1)\text{-form } \tilde{w} \quad w = d\tilde{w}$$

or function on $n=1$

for $\eta(x, u, v)$ if find F function st. $\eta = dF + g(u, v) q(x, u, v) du$

$$F = \frac{(u^3 - 1)uv}{(v^2 - xu)^3}$$

$$dF = \frac{\partial}{\partial u} F du + \frac{\partial}{\partial v} F dv = \left\{ \frac{\partial}{\partial u} - \frac{(u^2 - xv)}{v^2 - xu} \frac{\partial}{\partial v} \right\} F du = \dots$$

$$\eta(x, u, v) + dF \stackrel{(*)}{=} \frac{xu^4 + 5u^3v^2 + 3x^2u^2v + 4xuv^3 - v^5}{(v^2 - xu)^5} q(x, u, v) dx = 0$$

(*) check

Modular functions

$$(p'(\tau))^2 = 4p^3(\tau) - g_2 p(\tau) - g_3$$

$$\Delta = g_2^3 - 27g_3^2 \quad \text{homogeneous of degree 12}$$

$$g_2(\lambda, \omega_1, \lambda \omega_2) = \lambda^{-4} g_2(\dots)$$

$$g_3(\dots) = \lambda^{-6} g_3(\dots)$$

$$\tau = \frac{\omega_2}{\omega_1}, \quad \tau \in H$$

Klein's modular function $j(\tau)$ - homop. of degree 0

$$j(\omega_1, \omega_2) = \frac{g_2^3(\omega_1, \omega_2)}{\Delta(\omega_1, \omega_2)}; \quad j(\lambda \omega_1, \lambda \omega_2) = j(\omega_1, \omega_2)$$

$$j(\lambda, \tau) = j(\omega_1, \omega_2) \quad (\lambda = \frac{1}{\omega_1})$$

Theorem: $g_2(\tau), g_3(\tau), \Delta(\tau), j(\tau)$ - analytic in H .

Proof: (sketch)

$$\Delta(\lambda, \tau) \neq 0 \quad \text{in } H$$

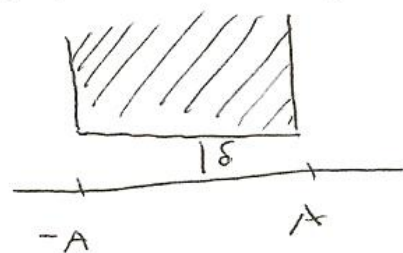
$$(\omega_1, \omega_2) \quad \tau = \frac{\omega_2}{\omega_1}$$

g_2, g_3 are given by $\sum_{\substack{(m,n) \neq (0,0) \\ (m,n) = -\infty}}^{\infty} \frac{1}{(m+n\tau)^\alpha}, \quad \alpha > 2$ up to constant factor

$$\tau = x + iy, \quad y > 0$$

$$\tau \in H$$

we want to show that series is absolutely convergent for
 $\forall$ fixed $\tau \in H$, uniformly in every strip $S = \{x + iy : |x| \leq A, y \geq \delta > 0\}$



we want to find const $M > 0$

$$M(A, \delta) \text{ s.t.}$$

$$\frac{1}{|m+n\tau|^\alpha} \leq \frac{M}{|m+ni|^\alpha}$$

$$\Leftrightarrow \exists \text{ const } K: |m+n\tau|^2 > K|m+ni|^\alpha \quad (1)$$

$$\Leftrightarrow (m+nx)^2 + (ny)^2 > K(m^2+n^2) \quad (1)$$

if $n=0$ then (1) holds $\forall K \in (0,1)$

if $n \neq 0$ $q = \frac{m}{n}$, to prove (1) it is equivalent to show

$$\text{that } (\exists K:) \frac{(q+x)^2 + y^2}{1+q^2} > K \quad (2)$$

In fact (2) holds for all q with $K = \frac{\delta^2}{1+(A+\delta)^2}$ if

$$|x| \leq A, y \geq \delta: \text{ why?}$$

$$\text{if } |q| \leq A+\delta \Rightarrow (2)$$

$$(q+x)^2 \geq 0$$

$$y^2 \geq \delta^2$$

$$\frac{(q^2+x)^2 + y^2}{1+q^2} \geq \frac{\delta^2}{1+(A+\delta)^2} = K$$

$$\text{if } |q| > A + \delta$$

$$\frac{|x|}{|q|} < \frac{|x|}{A+\delta} \leq \frac{A}{A+\delta} < 1$$

$$|1 + \frac{x}{q}| \geq 1 - |\frac{x}{q}| > 1 - \frac{A}{A+\delta} = \frac{\delta}{A+\delta}$$

$$|q+x| \geq \frac{|q|\delta}{A+\delta}$$

$$\Rightarrow \frac{(q+x)^2 + y^2}{1+q^2} > \frac{\delta^2 q^2}{(A+\delta)^2(1+q^2)} \quad (3)$$

$$\text{function } \frac{t}{1+t} \text{ is increasing } \Rightarrow \frac{q^2}{1+q^2} > \frac{(A+\delta)^2}{1+(A+\delta)^2}$$

when $q^2 > (A+\delta)^2$ so substituting this into (3) we get (2).

$$\omega_2' = a\omega_2 + b\omega_1$$

$$\omega_1' = c\omega_2 + d\omega_1$$

$$a, b, c, d \in \mathbb{Z}$$

$$ad - bc = 1$$

pair of periods (ω_1', ω_2') is equivalent to (ω_1, ω_2)
(the same fundamental parallelogram)

$$g_2(\omega_1', \omega_2') = g_2(\omega_1, \omega_2)$$

$$g_3(-''-) = g_3(-''-)$$

$$\Delta(-''-) = \Delta(-''-)$$

$$J(-''-) = J(-''-)$$

$$\tau' = \frac{\omega_2'}{\omega_1'} = \frac{a\tau + b}{c\tau + d} = \frac{a\omega_2 + b\omega_1}{c\omega_2 + d\omega_1}$$

$$\text{Im } \tau' = \text{Im } \frac{a\tau + b}{c\tau + d} = \frac{ad - bc}{|c\tau + d|^2}$$

$$\text{Im } \tau = \frac{\text{Im } \tau}{|c\tau + d|^2} \quad \begin{matrix} \tau \in H \Rightarrow \\ \tau' \in H \end{matrix}$$

$$\text{Def: } \tau' = \frac{a\tau + b}{c\tau + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$$

$$\det(A) = 1$$

unimodular transformation

$\Rightarrow$ those transformations form a group under composition

modular group Γ .

$J(\tau) = J(\frac{a\tau + b}{c\tau + d})$ Klein's function is invariant under the action of modular group Γ .

in particular, $J(\tau + 1) = J(\tau)$ (periodic with period 1)

$J(\tau)$ has a Fourier expansion

$$J(\tau) = \sum_{n=-\infty}^{\infty} a(n) e^{2\pi i n \tau}, \quad \tau \in H$$

Theorem: this series is absolutely convergent

Proof:

$$x = e^{2\pi i \tau}$$

$\tau \in \mathbb{H}$ is mapped to $x \in D = \{x \in \mathbb{C} : 0 < |x| < 1\}$



$\tau \rightarrow$ unique x

$x \mapsto \infty$ many $\tau \in \mathbb{H}$

$$e^{2\pi i \tau} = e^{2\pi i \tau'} \Leftrightarrow \tau' = \tau + n, \quad n \in \mathbb{Z}$$

is $f(x)$ analytic in D ?

$$f'(x) = \frac{d}{dx} J(\tau) = \frac{d}{d\tau} J(\tau) \frac{d\tau}{dx} = J'(\tau) / \frac{dx}{d\tau} = \frac{J'(\tau)}{2\pi i e^{2\pi i \tau}} \neq 0$$

J - periodic

f - analytic in punctured disc

$\Rightarrow f(x) = \sum_{-\infty}^{\infty} a_n x^n$ Laurent expansion, absolutely convergent.

In fact $a(-n) = 0 \quad \forall n \geq 1$
 $a(-1) = 12^{-3}$

(Remark)

$12^3 J(\tau)$ has integer coefficients in Fourier expansion

Fourier expansions of $g_2(\tau), g_3(\tau)$

$\sum \frac{1}{(m+n\tau)^k}$ Eisenstein series

$$g_2(\tau) = 60 \sum_{m,n \neq 0} \frac{1}{(m+n\tau)^4}$$

$$g_3(\tau) = 140 \sum \frac{1}{(\quad)^6}$$

Theorem: $\tau \in \mathbb{H}$

$$g_2(\tau) = \frac{4\pi^4}{3} \left(1 + 240 \sum_{k=1}^{\infty} \sigma_3(k) e^{2\pi i k \tau} \right)$$

$$g_3(\tau) = \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \sigma_5(k) e^{2\pi i k \tau} \right\}$$

$$\sigma_\alpha(k) = \sum_{d|k} d^\alpha$$

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n \tau}$$

$\tau(n)$ - integer's $\tau(1) = 1$
 $\tau(2) = -24$

Ramanujan's τ - functions

$$12^3 J(\tau) = e^{-2\pi i \tau} + 744 + \sum_{n=1}^{\infty} c(n) e^{2\pi i n \tau}$$

integers with interesting arithmetic properties

for instance, $(n+1)c(n) \equiv 0 \pmod{24}, \quad n \geq 1$

$$c(5n) \equiv 0 \pmod{25}$$

$$c(7n) \equiv 0 \pmod{7} \dots$$

Remarks:

$$\pi \frac{\cos \pi t}{\sin \pi t} = \pi \cot \pi t = \frac{1}{t} + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left(\frac{1}{t+m} - \frac{1}{m} \right)$$

meromorphic function with simple poles

partial fractions decomposition

$$x = e^{2\pi i \tau} \quad 0 < |x| < 1$$

$$\pi i \log \pi \tau = \pi i \log \frac{e^{2\pi i \tau} + 1}{e^{2\pi i \tau} - 1} = \pi i \frac{x+1}{x-1} = -\pi i \left(\frac{x}{1-x} + \frac{1}{1-x} \right) =$$

$$= -\pi i \left(\sum_{r=1}^{\infty} x^r + \sum_{r=0}^{\infty} x^r \right)$$

$$\frac{1}{\tau} + \sum_{\substack{m \neq 0 \\ m=-\infty}}^{\infty} \left(\frac{1}{\tau+m} - \frac{1}{m} \right) = -\pi i \left(1 + 2 \sum_{r=1}^{\infty} x^r \right)$$

$$\Rightarrow \text{differentiate} \quad -\frac{1}{\tau^2} - \sum \frac{1}{(\tau+m)^2} = (-2\pi i)^2 \sum_{r=1}^{\infty} r e^{2\pi i r \tau}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(m+n\tau)^4} = \frac{8\pi^4}{3} \sum_{r=1}^{\infty} r^3 e^{2\pi i r n \tau}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(m+n\tau)^6} = -\frac{8\pi^6}{15} \sum_{r=1}^{\infty} r^5 e^{2\pi i r n \tau}$$

$\Rightarrow g_2, g_3$ expansions

Theorem: Modular group $\Gamma = \langle T, S \rangle$

$$T\tau = \tau + 1$$

$$S\tau = -\frac{1}{\tau}$$

$$T \approx \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$S^2 = -Id$$

$\forall A \in \Gamma$

$$A = T^{n_1} S T^{n_2} S \dots S T^{n_k} \quad n_i \in \mathbb{Z}$$

maybe not uniquely
Exercise: Find this decomposition for

$$A = \begin{pmatrix} 4 & 9 \\ 11 & 25 \end{pmatrix}$$

$$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

$$AT^n = \begin{pmatrix} 4 & 9 \\ 11 & 25 \end{pmatrix} \cdot \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4n+9 \\ 11 & 11n+25 \end{pmatrix}$$

idea: chose n. s.t. $11(n+25) < 11$
 $n = -2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ multiplied by some power of T s.t.

we get
 $\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$ with $|\tilde{a}| < |\tilde{c}|$

$$AT^{-2} = \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}$$

this is
an algorithm

$$AT^{-2}S = \begin{pmatrix} 1 & -4 \\ 3 & -11 \end{pmatrix}$$

$$AT^{-2}ST^4 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$AT^{-2}ST^4S = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}$$

$$AT^{-2}ST^4ST^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = S$$

$$AT^{-2}ST^4ST^3 = S$$

$$\Rightarrow A = ST^{-3}ST^{-4}ST^2$$

Proof 0 thm by induction:

Let $c \geq 0$ in $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

if $c=0 \Rightarrow ad=1 \Rightarrow a=d=\pm 1$

$$A = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} 1 & \pm b \\ 0 & 1 \end{pmatrix} = T^{\pm b}$$

$$c=1 \Rightarrow b=ad-1$$

$$A = \begin{pmatrix} a & ad-1 \\ 1 & d \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = T^a S T^d$$

$c > 1$ $ad-bc=1$ induction $d=cq+r, 0 < r < c$

$$AT^{-q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -aq+b \\ c & r \end{pmatrix}$$

$$AT^{-q}S = \begin{pmatrix} -aq+b & -a \\ r & c \end{pmatrix} \quad \text{induction step} \\ = \text{product of } TS$$

Fundamental regions:

$\mathbb{H}/\Gamma$

G - subgroup of Γ

τ' - equivalent to τ in $\mathbb{H}$ if $\tau' = A\tau, A \in G$

$\Rightarrow$ orbits $G\tau = \{A\tau \mid A \in G\}$

Take 1 point from the orbit $\Rightarrow \forall \tau \in \mathbb{H}$

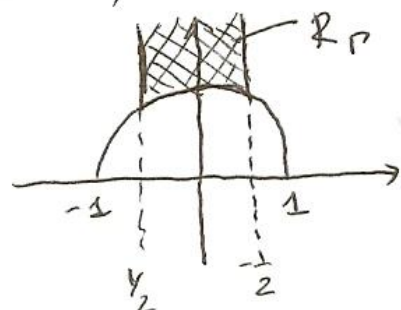
$\Rightarrow$ fundamental region R_G of G

$\bullet \forall \tau_1, \tau_2 \in R_G \Rightarrow \tau_1 \not\sim \tau_2$
(is not equivalent)

$\bullet \forall \tau \in \mathbb{H} \exists \tau_3 \in R_G$ s.t. $\tau = A\tau_3$ for some $A \in G$

fundamental region of Γ consists of $\tau \in \mathbb{H}$ s.t.

$$|\tau| > 1, \quad |\tau + \bar{\tau}| < 1$$



$$\tau = u + iv, \quad v > 0$$



this line is transformed to this line

Def: Modular function f

we call a function f modular if f is meromorphic in $\mathbb{H}$

$$f(A\tau) = f(\tau), \quad A \in \Gamma$$

$$f(\tau) = \sum_{n=-m}^{\infty} a(n) e^{2\pi i n \tau} \quad m \in \mathbb{Z} \quad (\text{for some})$$

$t = i^\infty$ pole of order m for some $m \in \mathbb{Z}$

Example: $J(z)$ has a 1st order pole at $i\infty$

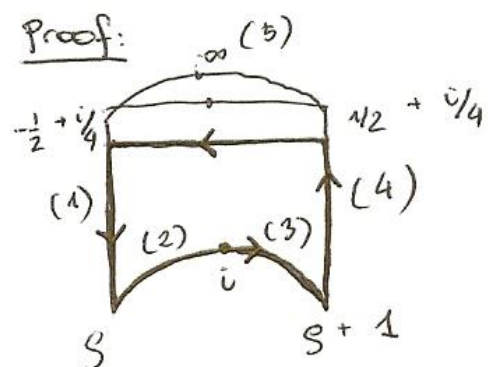
Next time:

$\forall$ modular f is a rational function of J (proof later on)

Th: f - modular, $f \neq 0$

$\Rightarrow$ in the closure of R_P # poles = # zeros

Proof:



$$\begin{aligned} (1) &\equiv (4) \\ (2) &\equiv (3) \\ S &= e^{2\pi i/3} \end{aligned}$$

zeros - # poles inside R_P

$$Z - P = \frac{1}{2\pi i} \int_{\partial R_P} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \left(\int_{(1)} + \int_{(2)} + \int_{(3)} + \int_{(4)} + \int_{(5)} \right) = \frac{1}{2\pi i} \int_{(5)} \frac{f'(z)}{f(z)} dz = (*)$$

$$u = Sz = -1/z$$

$$\frac{f'(z)}{f(z)} dz = \frac{f'(u)}{f(u)} du$$

take $x = e^{2\pi i z}$ $z = u + iM$

$$x = e^{2\pi i(u + iM)} = e^{-2\pi M} e^{2\pi i u}$$

$\Rightarrow x \in K$ circle of radius $e^{-2\pi M}$ around 0.



negative direction

Points above line (5) are mapped inside K

$\Rightarrow x=0$ is either a zero or a pole of $F(x) = f(z)$

$$f(z) = \frac{a-m}{x^m} + \dots \quad f'(z) = F'(x) \frac{dx}{dz}, \quad \frac{f'(z)}{f(z)} dz = \frac{F'(x)}{F(x)} dx$$

$$(*) = -\frac{1}{2\pi i} \oint_K \frac{F'(x)}{F(x)} dx = -(Z_F - P_F)$$

number of zeros and poles of F inside K

if $x=0$ a pole of order $m > 0$ $Z_F = 0$, $P_F = m$, $P_F - Z_F = m$

$$Z = P + m$$

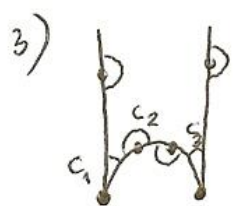
if $x=0$ is a zero of order n $\Rightarrow m = -n$

$$P_F = 0 \quad Z_F = n, \quad Z + n = P$$

2) if there are zeros or poles on ∂R_P

modification of the path





if f has a zero or a pole at vertex

$$Z - P = \frac{1}{2\pi i} \left(\int_{c_1} + \int_{c_2} + \int_{c_3} + \int_{\frac{1}{2} + iM}^{\frac{1}{2} + iM} \right) \frac{f'(z)}{f(z)} dz$$

if x^{-m} - lowest power of $f(z)$ in Laurent expansion $x=0$,
 $x = e^{2\pi i \tau}$

$$Z - P = \frac{1}{2\pi i} \left(\int_{c_1} + \int_{c_2} + \int_{c_3} \right) \frac{f'(z)}{f(z)} dz + m$$

Near vertex s

$$f(z) = (z-s)^k g(z), \quad k \in \mathbb{Z}, \quad g(s) \neq 0$$

$$c_1: z-s = re^{iQ} \quad \text{small } Q \in (d, \pi/2)$$

$$\frac{f'(z)}{f(z)} = \frac{k}{z-s} + \frac{g'(z)}{g(z)}$$

$$\frac{1}{2\pi i} \int_{c_1} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{\pi/2}^{\alpha} \left(\frac{k}{re^{iQ}} + \frac{g'(s+re^{iQ})}{g(s+re^{iQ})} \right) re^{iQ} i dQ =$$

$$= - \frac{k \left(\frac{\pi}{2} - \alpha \right)}{2\pi} + \frac{r}{2\pi} \int_{\pi/2}^{\alpha} \frac{g'}{g} e^{iQ} dQ$$

$\downarrow r \rightarrow 0 \quad \frac{\pi}{3}$
 $\downarrow r \rightarrow 0 \quad 0$

$$\Rightarrow \lim_{r \rightarrow 0} \frac{1}{2\pi i} \int_{c_1} \frac{f'(z)}{f(z)} dz = -\frac{k}{6}$$

similar for c_3 , $f(z) = (z-i)^l h(z)$, $h(i) \neq 0$

$$\lim_{r \rightarrow 0} \frac{1}{2\pi i} \int_{c_2} \frac{f'(z)}{f(z)} dz = -\frac{l}{2}$$

$$Z - P = m - \frac{k}{6} - \frac{k}{6} - \frac{l}{2}$$

depending on whether k or l is negative or positive

Corollary: f -modular
 f -bounded in H

$$\Rightarrow f = \text{const.}$$

18.05.2011

Th: $J(s) = 0$ triple zero
 $J(i) = 1$ double point
 $J(i_{\infty})$ - 1-st order pole

Proof: $g_2(s) = 0$ because $s^3 = 1$ $s^2 + s + 1 = 0$

$$\frac{1}{10} g_2(s) = \sum \frac{1}{(m+ng)^4} = \sum_{m,n} \frac{1}{(mg^3+ng)^4} = \frac{1}{g^4} \sum \frac{1}{(mg^2+n)^4} =$$

$$= \frac{1}{S} \sum_{m=1}^{\infty} \frac{1}{(n-m-m_S)^4} = \frac{1}{S} \sum_{M=N}^{\infty} \frac{1}{(N+M_S)^4} = \frac{1}{60S} g_2(S)$$

$$g_3(i) = 0 \quad \text{by analogue}$$

$$j(i) = \frac{g_2^3(i)}{g_3^3(i)}$$

Theorem: 1) $\forall$ rational function of J is a modular function
 2) $\forall$ modular function f can be expressed as a rational function of J .

Proof: 1) clear

2) $x_1, \dots, x_n$ - zeros of f
 $p_1, \dots, p_n$ - poles

$$g(\tau) \stackrel{\text{def}}{=} \prod_{k=1}^n \frac{j(\tau) - j(z_k)}{j(\tau) - j(p_k)} \quad \text{- some zeroes and poles as in } f \text{ in } R_{\Gamma}$$

f/g - constant

$$(p'(z))^2 = 4p^3(z) - g_2 p(z) - g_3 \quad (*)$$

w_1, w_2 - periods of $p(z)$

$$\begin{cases} * \\ * \\ * \end{cases} \begin{cases} g_2 = g_2(w_1, w_2) = 60 \sum_{m,n \neq 0} \frac{1}{(mw_1 + nw_2)^2} \\ g_3 = g_3(w_1, w_2) = 140 \sum \frac{1}{(mw_1 + nw_2)^4} \end{cases}$$

$$g_2^3 - 27g_3^2 \neq 0 \quad (*)$$

Question: inverse problem:

given $g_2, g_3 \in \mathbb{C}$ satisfying $(*)$ can we find w_1, w_2 s.t.
 $(**) holds. (in other words, can we solve $(***)$ for w_1, w_2)?$

Theorem: given two complex numbers a_2, a_3 s.t. $a_2^3 - 27a_3^2 \neq 0$
 then $\exists w_1, w_2 \in \mathbb{C} \quad \text{Im} \frac{w_2}{w_1} \neq 0$ s.t. $g_2(w_1, w_2) = a_2, g_3(w_1, w_2) = a_3$.

case 1: $a_2 = 0, a_3 \neq 0$

case 2: $a_3 = 0, a_2 \neq 0$

case 3: $a_2 a_3 \neq 0$

case 1: Take $g = e^{2\pi i/3}$

$$w_1 \stackrel{\text{def}}{=} \frac{g_3(1, g)}{a_3} \quad w_2 \stackrel{\text{def}}{=} g w_1$$

$$g_2(w_1, w_2) = g_2(w_1, w_2, g) = \frac{1}{w_1^4} g_2(1, g) = 0 = a_2$$

$$g_3(w_1, w_2) = g_3(w_1, w_2, g) = \frac{1}{w_1^6} g_3(1, g) = a_3 \neq 0$$

our choice of ω_1, ω_2 - ok

Case 2:

Take $\omega_1^4 = \frac{g_2(1, i)}{a_2}$

$\omega_2 = i\omega_1$

$g_2(\omega_1, \omega_2) = g_2(\omega_1, i\omega_1) = \frac{1}{\omega_1^4} g_2(1, i) = a_2$

$g_3(\omega_1, \omega_2) = g_3(\omega_1, i\omega_1) = \frac{1}{\omega_1^6} g_3(1, i) = 0 = a_3$

Case 3 general:

Choose $\tau \in \mathbb{C}$ with $\text{Im } \tau > 0$ $J(\tau) = \frac{a_2^3}{a_2^3 - 27a_3^2} \neq 0$

$\frac{J(\tau) - 1}{J(\tau)} = \frac{27a_3^2}{a_2^3} \quad (*)$ take $\omega_1^2 = \frac{a_2}{a_3} \frac{g_3(1, \tau)}{g_2(1, \tau)}$

$\omega_2 = \tau\omega_1$

$\frac{g_2(\omega_1, \omega_2)}{g_3(\omega_1, \omega_2)} = \frac{\omega_1^{-4} g_2(1, \tau)}{\omega_1^{-6} g_3(1, \tau)} = \frac{\omega_1^2 g_2(1, \tau)}{g_3(1, \tau)} = \frac{a_2}{a_3}$

$g_3(\omega_1, \omega_2) = \frac{a_3}{a_2} g_2(\omega_1, \omega_2) \quad (****)$

$\frac{J(\tau) - 1}{J(\tau)} = \frac{27 g_3^2(\omega_1, \omega_2)}{g_2^3(\omega_1, \omega_2)} = \frac{27 \left(\frac{a_3}{a_2}\right)^2 g_2^2(\omega_1, \omega_2)}{g_2^3(\omega_1, \omega_2)} = \frac{27 a_3^2}{a_2^2 g_2(\omega_1, \omega_2)} \stackrel{(*)}{=} \frac{27 a_3^2}{a_2}$

$\Rightarrow g_2(\omega_1, \omega_2) = a_2$

from $(****)$ $g_3(\omega_1, \omega_2) = a_3$

Congruence subgroups
→ number theory

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$\det A, B = 1$
 $a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{Z}$

we say that

$A \equiv B \pmod n$

if $\begin{matrix} a \equiv \alpha \pmod n \\ b \equiv \beta \pmod n \\ c \equiv \gamma \pmod n \\ d \equiv \delta \pmod n \end{matrix}$

equivalence relation

$A_1 \equiv A_2, B_1 \equiv B_2 \Rightarrow A_1 B_1 \equiv A_2 B_2$

$A_1^{-1} \equiv A_2^{-1} \pmod n$

$A \equiv B \Leftrightarrow AB^{-1} \equiv I \pmod n$

$\Gamma(n)$ - set of all matrices in Γ congruent modulo n to identity

$$j(\tau) = 12^3 J(\tau) = \frac{1}{x} + \sum_{n=0}^{\infty} c(n)x^n, \quad x = e^{2\pi i \tau}$$

$$c(2n) \equiv 0 \pmod{2^{11}}$$

$$c(3n) \equiv 0 \pmod{3^5}$$

$$c(5n) \equiv 0 \pmod{5^2}$$

$$c(7n) \equiv 0 \pmod{7}$$

$$f_5(\tau) = \sum_{n=0}^{\infty} c(5n)x^n \quad \text{invariant with respect to subgroup}$$

$$c \equiv 0 \pmod{q} \quad (\text{prime number or power of a prime})$$

$$\Gamma_0(q) \subset \Gamma$$

$$\Gamma_0(q) \text{ finite index in } \Gamma$$

$\forall$ element in Γ can be expressed represented in terms of elements in $\Gamma_0(q)$

→ fundamental region $\Gamma_0(p)$

→ generators of $\Gamma_0(p)$

→ functions invariant $\Gamma_0(p)$

Fix p -prime

Theorem: $S\tau = -1/\tau$ $T\tau = \tau + 1$ generators of Γ : $\forall v \in \Gamma, v \notin \Gamma_0(p)$
 $\exists \bar{P} \in \Gamma_0(p) \exists k \in \mathbb{N}$

$$0 \leq k < p \quad \text{s.t.} \quad v = \bar{P} S T^k$$

Proof:

$$\text{Let } v = \begin{pmatrix} A & B \\ c & D \end{pmatrix} \quad c \not\equiv 0 \pmod{p}$$

$$\text{we want to find } P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \begin{matrix} c \equiv 0 \\ \not\equiv 0 \end{matrix} \pmod{p}$$

$$\text{s.t.} \quad \begin{pmatrix} A & B \\ c & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} S T^k$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

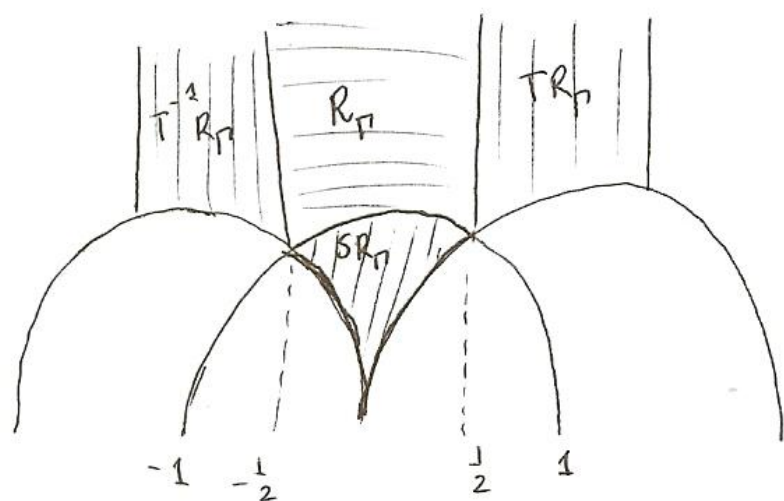
$$T^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

$$P S T^k = \begin{pmatrix} b & -a+kb \\ a & -c+ke \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A & B \\ c & D \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix}^{-1} = \begin{pmatrix} A & B \\ c & D \end{pmatrix} \begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} kA-B & A \\ kC-D & C \end{pmatrix}$$

$$\text{choose } k : kC \equiv D \pmod{p} \quad 0 \leq k < p$$

$$\text{fundamental region of } \Gamma_0(p) : R_p \cup \bigcup_{k=0}^{p-1} S T^k(R_p) \quad , p\text{-prime}$$



Theorem: $\forall p > 3$ p -prime

$\Gamma_0(p)$ has $2 \left[\frac{p-1}{12} \right] + 3$ generators and they are selected

from elements $T, V_1, \dots, V_{p-1}$

$$T\tau = \tau + 1, \quad S\tau = -1/\tau, \quad V_k = ST^k ST^{-k}, \quad S = \begin{pmatrix} k' & 1 \\ -(kk'+1) & -k \end{pmatrix}$$

$$kk' \equiv -1 \pmod{p}$$

Exercise: $\Gamma_0(2) = \langle S, V_1 \rangle$; $V_1 = ST^1 ST^{-1} S$
 $= \langle T, V_1 \rangle$

$$\Gamma_0(3) = \langle T, V_2 \rangle$$

functions automorphic under $\Gamma_0(p)$

f : meromorphic in H

$$f(A\tau) = f(\tau), \quad A \in \Gamma_0(p)$$

$$A\tau = \frac{a\tau + b}{c\tau + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Theorem: f -automorphic under $\Gamma_0(p)$, bounded in $H \Rightarrow$ constant.

Th: f -modular function (or automorphic under Γ), p -prime $\Rightarrow$

$$f_p(\tau) \stackrel{\text{def}}{=} \frac{1}{p} \sum_{\lambda=0}^{p-1} f\left(\frac{\tau + \lambda}{p}\right) \text{ - automorphic under } \Gamma_0(p)$$

$$\text{also } f(\tau) = \sum_{n=-m}^{\infty} a(n) e^{2\pi i n \tau}$$

$$\Rightarrow f_p(\tau) = \sum_{n=\lfloor \frac{m}{p} \rfloor}^{\infty} a(np) e^{2\pi i n \tau}$$

$$T\tau = \tau + 1$$

$$S\tau = -\frac{1}{\tau}$$

$$T \in \Gamma_0(p)$$

$$(S \notin \Gamma_0(p))$$

$$f_p(T\tau) = f_p(\tau)$$

$$f_p\left(-\frac{1}{\tau}\right) = f_p(\tau) + \frac{1}{p} f(p\tau) - \frac{1}{p} f\left(\frac{\tau}{p}\right)$$

$$\Delta(\tau) = g_2^3 - 27g_3^2 = (24)^{12} \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n \tau}$$

$$\Delta\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{12} \Delta(\tau) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

$$\Delta(\tau + 1) = \Delta(\tau)$$

$$\Delta\left(-\frac{1}{\tau}\right) = \tau^{1/2} \Delta(\tau)$$

$$q \in \mathbb{N}$$

$$\varphi(\tau) = \frac{\Delta(q\tau)}{\Delta(\tau)}, \quad \tau \in H$$

$$\text{automorphic under } \Gamma_0(q)$$

$$\left(1 + \sum_{n=1}^{\infty} b_n x^n\right), \quad x = e^{\frac{2\pi i \tau}{q}}, \quad b_n \in \mathbb{Z}$$

$$\varphi(\tau) = x^{q-1}$$

Prove that

$$\varphi\left(-\frac{1}{q\tau}\right) = \frac{1}{q^{12}\varphi(\tau)}$$

and hence, $\varphi(\tau) \xrightarrow{\tau \rightarrow 0} \infty$

$$\varphi\left(-\frac{1}{q\tau}\right) = \frac{\Delta\left(q\left(-\frac{1}{q\tau}\right)\right)}{\Delta\left(-\frac{1}{q\tau}\right)} = \frac{\Delta\left(-\frac{1}{\tau}\right)}{\Delta\left(-\frac{1}{q\tau}\right)} = \frac{\tau^{12}\Delta(\tau)}{q^{12}\tau^{12}\Delta(q\tau)} = \frac{1}{q^{12}\varphi(\tau)}$$

$$\alpha = \frac{1}{q-1}$$

$$\Phi(\tau) = \varphi^\alpha(\tau) = q^{-1}\sqrt{\varphi(\tau)} \leadsto 1\text{-st order zero at } i_\infty$$

$$\frac{1}{\Phi(\tau)} = \frac{1}{x} + \text{power series in } x \text{ with integer coeff.}$$

$\Phi(\tau)$ - plays similar role as J in modular functions

$$\varphi(v\tau) = \varphi(\tau) \quad \forall v \in \Gamma_0(q)$$

$$\Phi(v\tau) = \varepsilon \Phi(\tau), \quad \varepsilon^{q-1} = 1$$

if $q = 2, 3, 5, 7, 13 \Rightarrow \varepsilon = 1$ and $\Phi(i)$ - automorphic for $\Gamma_0(q)$

Modular forms:

Examples: $\Delta, G_{2k}(\tau) = \sum_{n \neq 0} \frac{1}{(m+n\tau)^{2k}}, \quad k \geq 2$

satisfy $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-r} f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, r \in \mathbb{Z}$

for $\Delta(\tau) \quad r=12$
 $G_{2k}(\tau) \quad r=2k$

periodic with period 1

Def: Modular form of weight $k \in \mathbb{Z}$ is a function f if

Entire 1) analytic in $\mathbb{H}$

2) $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

3) $f(\tau) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n \tau}$

Example:

J - nonentire modular form of weight 0

$f \equiv \text{const} \neq 0$ modular form of weight 0 (entire)

entire modular form of weight 0 is only a constant function!

$\exists$ nonconstant entire modular forms only if k even, $k \geq 4$

They can be expressed in terms of G_4, G_6

Theorem:

f - entire modular form of weight k , $f \neq \text{const}$, f has N zeros in R_Γ omitting vertices; $\tau = g$ is a zero of order $N(g)$

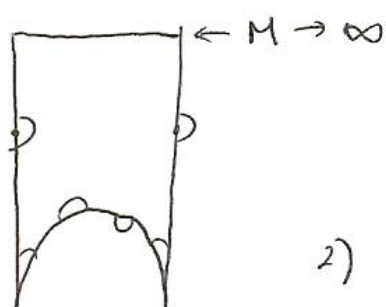
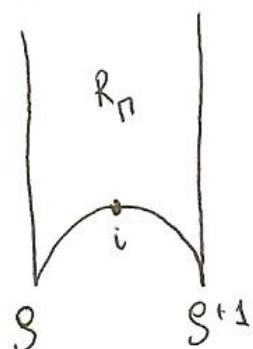
$\tau = i$ - zero of order $N(i)$

$\tau = i_\infty$ - zero of order $N(i_\infty)$

$$\Rightarrow k = 12N + 6N(i) + 4N(g) + 12N(i_\infty)$$

(hence, k - even)

Proof:



$$N = \frac{1}{2\pi i} \int_{\partial R_n} \frac{f'(\tau)}{f(\tau)} d\tau \quad (\text{no poles - entire})$$

1) modify ∂R_n

2) last time for modular functions we had

$$N = -\frac{1}{2}N(i) - \frac{1}{3}N(g) - N(i_\infty)$$

$+\frac{k}{12}$ - comes from $(c\tau+d)^k$
 $f(A\tau) = (c\tau+d)^k f(\tau)$

$$(f'(A\tau))' = (c\tau+d)^k f'(\tau) + kc(c\tau+d)^{k-1} f(\tau)$$

$$\frac{f'(A\tau)A'(\tau)}{f(A\tau)} = \frac{f'(\tau)}{f(\tau)} + \frac{kc}{c\tau+d}$$

in $\int_{\partial R_n}$ cancel out



$$\frac{1}{2\pi i} \int_{A(\gamma)} \frac{f'(u)du}{f(u)} = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(\tau)}{f(\tau)} d\tau + \frac{1}{2\pi i} \int_{\gamma} \frac{k}{c\tau+d} d(c\tau+d)$$

$$\frac{k}{2\pi i} \int_{\gamma} \frac{d\tau}{\tau} = \frac{k(\log i - \log g)}{2\pi i} = \frac{k}{12}$$

$$g = e^{2\pi i/3}$$

$$i = e^{2\pi i/4}$$

(JUST SKETCH OF THE PROOF)

Theorem: only modular forms of weight $k=0$ are constant functions

1) The only modular forms of weight $k=0$ are constant functions

2) k -odd, $k < 0$ or $k = 2 \Rightarrow$ only entire form of weight k

is zero function

3) $\forall$ nonconstant entire modular function has weight k even, $k \geq 4$

4) $\forall$ entire function of weight k can be expressed in terms of G_{2k} and powers of $\Delta(\tau)$

$$\Delta = g_2^3 - 27g_3^2 = (60G_4)^3 - 27(140G_6)^2$$

$$G_0(\tau) = 1 \quad f = \sum_{r=0, k-12r \neq 2} a_r G_{k-2r} \Delta^r, \quad a_r \in \mathbb{C}$$

$$f = \sum_{\substack{a,b, a \geq 0, b \geq 0 \\ 4a+6b=k}} c_{ab} G_4^a G_6^b$$

c_{ab} - complex

5) $z_1, \dots, z_N$ - zeroes of f in $\overline{\mathbb{R}_p}$ | vertices

$N(g), N(i), N(i_\infty)$ - orders of zeros at verticity

$$\Rightarrow f(\tau) = c G_4(\tau)^{N(g)} G_6(\tau)^{N(i)} \Delta(\tau)^{N(i_\infty)} \Delta(\tau)^N \prod_{k=1}^N (J(\tau) - J(z_k))$$

some constant

Proof:

$$5) g(\tau) := \prod_{k=1}^N (J(\tau) - J(z_k))$$

modular function $z_1, \dots, z_N$ - zeros in $\overline{\mathbb{R}_p}$
pole of order N at i_∞

Δ has a first order 0 at i_∞

$\Delta^N g$ - entire modular form of weight $12N$
with zeros only at $z_1, \dots, z_N$

$$h = G_4^{N(g)} G_6^{N(i)} \Delta^{N(i_\infty)} \Delta^N g$$

same zeros as f modular form of weight

$$k = 4N(g) + 6N(i) + 12N(i_\infty) + 12N$$

$\Rightarrow \frac{f}{h}$ - entire form of weight 0 - const

25.05.2011

Modular spaces of elliptic curves and hypergeometric equation

$\mathbb{P}^1$

$$x_1 = [u_{11} : u_{21}]$$

$$x_1 \neq x_2 = [u_{12} : u_{22}]$$

$$y_1 = [v_{11} : v_{21}]$$

$$y_1 \neq y_2 = [v_{12} : v_{22}]$$

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = P \cdot \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

$\det \neq 0$ $\det \neq 0$

$$P \in GL(2, \mathbb{C})$$

if we have 3 distinct points they can be transformed

to $0, 1, \infty$.

$$A = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{pmatrix}$$

Projective form
of $0, 1, \infty$

$$\begin{matrix} 0 & 1 & \infty \\ \downarrow & \downarrow & \downarrow \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{matrix}$$

$$[u:v] = [1, \frac{u}{v}]$$

Multiply A from left to get

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & v \end{pmatrix}, \text{ multiply this by } \begin{pmatrix} 1 & 0 \\ 0 & 1/v \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1/v & 1 \end{pmatrix}$$

if 4 points minors $D(i,j) \neq 0$
 $i \neq j$

$$A = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}$$

$$A \sim \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \lambda \end{pmatrix} \quad z_1 = \frac{x_{11}}{x_{21}}, \dots, z_4 = \frac{x_{14}}{x_{24}} \quad \text{normal form}$$

$$\lambda(x) = \frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_3)(z_2 - z_4)} \quad \lambda\text{-invariant}$$

Another realization is the following (if all points are treated equally).

$$f: x \mapsto \text{points in } \mathbb{P}^2 \text{ given by } (D(1,2) D(3,4) : D(1,3) D(2,4) : D(1,4) D(2,3)) = (y_1 : y_2 : y_3) \quad y_1, y_2, y_3 \neq 0$$

$$y_1 - y_2 + y_3 = 0$$

for normal form

$$(y_1 : y_2 : y_3) = (1(\lambda-1) : 1(-1) : \lambda(-1)) = (\lambda-1 : -1 : -\lambda)$$

$X(4)$ - configuration space of 4 points in $\mathbb{P}^1$ (ordered)

$X\{4\}$ - ———— || ———— not ordered

$X\{k,n\} := X(k,n)/S_n$ - configuration space of ordered points in $\mathbb{P}^{k-1}$

Take normal form

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \lambda \end{pmatrix} \text{ in } X\{4\} \quad \text{same as} \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & \lambda \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & \lambda \\ 0 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \lambda \end{pmatrix}$$

↑
replaced

↓ the same as

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & \lambda \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & \lambda-1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1-\lambda \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1-\lambda \end{pmatrix}$$

we get in $(2,4)$ element $\lambda \quad \frac{1}{\lambda} \quad 1-\lambda \quad \frac{1}{1-\lambda} \quad \frac{\lambda}{\lambda-1} \quad \frac{\lambda-1}{\lambda}$ in the normal form.

we look for a rational function invariant under all these transformations

$$j(\lambda) = \frac{4}{27} \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2}$$

$$\begin{cases} \lambda = \frac{1}{\lambda} \Leftrightarrow \lambda^2 = 1 \Rightarrow \lambda = 1, -1 \text{ fixed points} \\ \lambda = 1-\lambda \Leftrightarrow 2\lambda = 1 \Rightarrow \lambda = 1/2, \infty \\ \frac{1}{1-\lambda} = \lambda \Leftrightarrow \lambda^2 - \lambda + 1 = 0 \end{cases}$$

$\lambda \rightarrow j(\lambda)$ gives a realization of $X\{4\} = X(4)/S_4$

For $X(4)$ we had also a realization

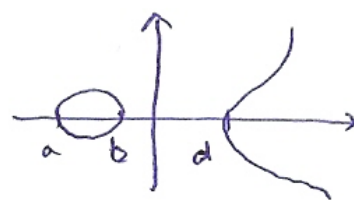
$$x \rightarrow [D(1,2) D(3,4) : D(1,3) D(2,4) : D(1,4) D(2,3)] \in \mathbb{P}^2$$

we want an invariant rational function f of $D(i,j) D(p,q)$ with respect to S_4 for $X\{4\}$

$$\frac{[(D(12)D(34))^2 + (D(13)D(24))^2 + (D(14)D(23))^2]^3}{(D(12)D(34))^2 (D(13)D(24))^2 (D(14)D(23))^2} = \frac{27}{4} \cdot 2^3 j = j$$

$$y^2 = (x-a)(x-b)(x-c) \quad d = \infty$$

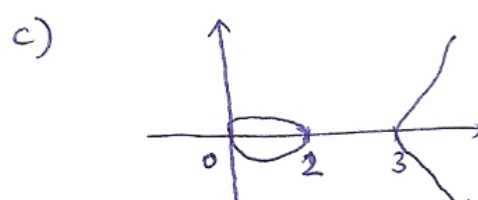
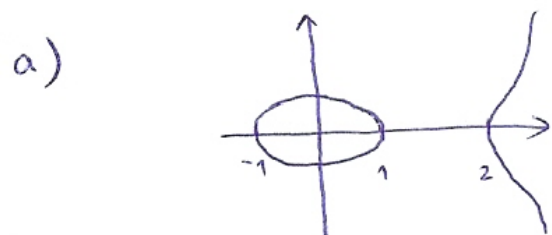
$$y^2 = (x-a)(x-b)(x-c)(x-d)$$



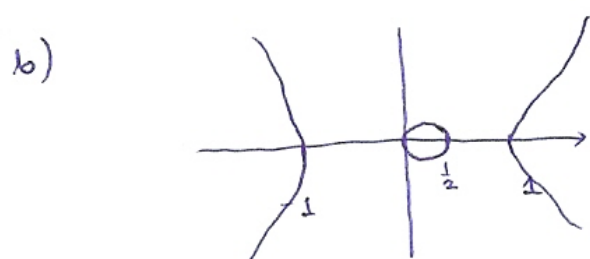
$$a) y^2 = (x+1)(x-1)(x-2)$$

$$b) y^2 = (x+1)x(x-1)(2x-1)$$

$$c) y^2 = x(x-2)(x-3)$$



$$a) \rightarrow c) \quad (x, y) \rightarrow (x-1, y)$$



$$x \rightarrow \frac{1}{x}$$

$$y \rightarrow \frac{y}{x^2}$$

$$\text{from } a) \rightarrow b) \quad \frac{y^2}{x^4} = \left(\frac{1+x}{x} \right) \left(\frac{1-x}{x} \right) \left(\frac{2-x}{x} \right)$$

when (a, b, c, d) in $y^2 = (x-a)(x-b)(x-c)(x-d)$ and (e, f, g, h) in $y^2 = (x-e)(x-f)(x-g)(x-h)$ give some elliptic curve?

$$\{a, b, c, d\} \in X(4) \quad (a, b, c, d) \rightarrow \frac{(a-d)(b-c)}{(a-c)(b-d)} \quad \lambda\text{-invariant}$$

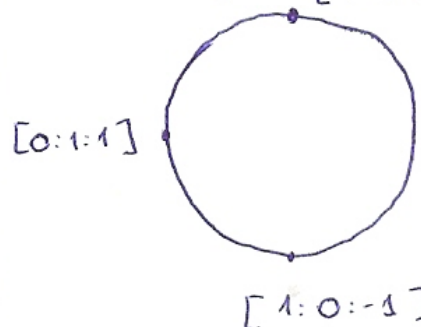
$$X(4) \simeq \mathbb{C} \setminus \{0, 1\}$$

$$\mathbb{P}^1 \setminus \{0, 1, \infty\}$$

$$(a, b, c, d) \rightarrow ((a-d)(b-c) : (a-c)(b-d) : (a-b)(d-c)) \in \mathbb{P}^2$$

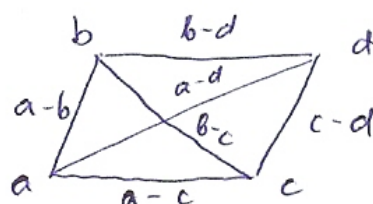
$$y_1 - y_2 + y_3 = 0$$

$$\mathbb{P}^2 \quad [1:1:0]$$



$$y_1 - y_2 + y_3 = 0 \mid y_1 y_2 y_3 = 0$$

$\Rightarrow$ isomorphism $X(4) \simeq$ line in $\mathbb{P}^2$ minus 3 points



$E(i) = E(\omega_1, \omega_2)$ - elliptic curve

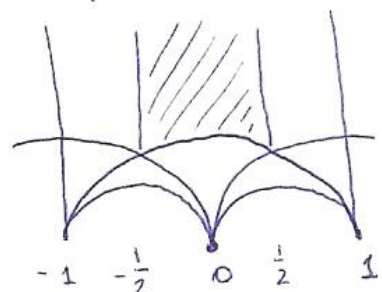
$$\tau = \frac{\omega_2}{\omega_1}$$

2 elliptic curves (ω_1, ω_2) and (ω_1', ω_2') are isomorphic

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad a, b, c, d \in \mathbb{Z} \quad ad - bc = 1$$

$$\tau' = g\tau, \quad g \in \text{PSL}(2, \mathbb{Z})$$

isomorphic classes of elliptic curves $\mathbb{H} / \text{PSL}(2, \mathbb{Z})$



fundamental domain

$$\mathbb{H} / \text{SL}(2, \mathbb{Z}) \cong X(4)$$

$$j(\lambda) = \frac{4}{27} \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1 - \lambda)^2}$$

zeroes of $4x^3 - g_2x - g_3$

$$e_3 = p\left(\frac{\omega_1}{2} + \frac{\omega_2}{2}\right)$$

$$e_1 = p\left(\frac{\omega_1}{2}\right), \quad e_2 = p\left(\frac{\omega_2}{2}\right)$$

$$\lambda = \lambda(\omega_1, \omega_2) = \frac{e_1 - e_2}{e_1 - e_3}$$

$$j(\lambda) = \frac{4}{27 \cdot 23} \frac{\{(e_1 - e_2)^2 + (e_2 - e_3)^2 + (e_3 - e_1)^2\}^3}{((e_1 - e_2)(e_2 - e_3)(e_3 - e_1))^2} = \frac{g_2^3}{g_2^3 - 27g_3^2}$$

$$e_1 + e_2 + e_3 = 0$$

$$e_1 e_2 + e_2 e_3 + e_3 e_1 = -\frac{g_2}{4}$$

$$e_1 e_2 e_3 = \frac{g_3}{4}$$

If we have 2 elliptic curves, we want order:

$$\frac{\omega_1}{2} \rightarrow \frac{\omega_1'}{2}$$

$$\frac{\omega_2}{2} \rightarrow \frac{\omega_2'}{2}$$

$$\mathbb{H} / \Gamma(2) \quad \Gamma(2) = \{g \in \text{SL}(2, \mathbb{Z}) : g \equiv I_2 \pmod{2}\}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \begin{array}{l} a, d - \text{odd} \\ c, b - \text{even} \end{array}$$

$$\Gamma(2) = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, I_2 \right\rangle$$

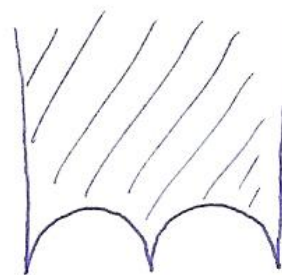
fundamental domain $\mathbb{H} / \Gamma(2)$

$$\lambda = \frac{e_1 - e_2}{e_1 - e_3}$$

$$\lambda: \mathbb{H} / \Gamma(2) \xrightarrow{\sim} \mathbb{C} / \langle 9, 13 \rangle \cong X(4) \quad - \text{line in } \mathbb{P}^2 \text{ 13 points}$$

$$\varphi: E(\tau) \xrightarrow{\text{emb}} \mathbb{P}^3$$

$$(\theta_{0,0}(2z, \tau) : \theta_{0,1}(2z, \tau) : \theta_{1,0}(2z, \tau) : \theta_{1,1}(2z, \tau))$$



$$\varphi(\tau): \begin{cases} \Theta_{00}^2 \Theta_{00}(z) - \Theta_{01}^2 \Theta_{01}(z)^2 - \Theta_{10}^2 \Theta_{10}(z)^2 = 0 \\ \Theta_{10}^2 \Theta_{01}(z)^2 - \Theta_{01}^2 \Theta_{10}(z)^2 - \Theta_{00}^2 \Theta_{11}(z)^2 = 0 \end{cases} \quad \text{when } \Theta_{jk} = \Theta_{jk}(\tau)$$

Riemann's relations

$$\begin{cases} ax^2 + by^2 + cz^2 = 0 \\ ey^2 + bz^2 + dw^2 = 0 \end{cases} \quad \text{in } \mathbb{P}^3 - \text{intersection of 2 quadrics curves}$$

Jacobi's identity $\Theta_{00}(\tau)^4 \neq \Theta_{01}(\tau)^4 - \Theta_{10}(\tau)^4 = 0$

$$\tau \rightarrow (y_1: y_2: y_3) \in \mathbb{P}^2 = (-\Theta_{00}^4(\tau): \Theta_{01}^4(\tau): \Theta_{10}^4(\tau))$$

$$J: \mathbb{H} / \mathrm{SL}(2, \mathbb{Z}) \xrightarrow{\sim} \mathbb{X}(4)$$

$$\tau \rightarrow J(\tau) = \text{const} \cdot \frac{(\Theta_{00}^8(\tau) + \Theta_{01}^8(\tau) + \Theta_{10}^8(\tau))^3}{\Theta_{00}^8 \Theta_{01}^8 \Theta_{10}^8(\tau)}$$

$$\begin{array}{ccc} \mathbb{C} / \langle \tau \rangle \simeq \mathbb{X}(4) & \xleftarrow{\lambda} & \mathbb{H} / \Gamma(2) \\ & \xrightarrow{\psi} & \\ \text{realization} & & \text{theta functions} \\ \searrow & & \swarrow \\ D(ij) & & \text{lin } J \subset \mathbb{P}^2 \end{array}$$

$$x(1-x)u'' + \{c - (a+b+1)x\}u' - abu = 0$$

hypergeometric equation

$$u = u(x)$$

0, 1, ∞ - singular points

$$(a, b, c) = \left(\frac{1}{2}, \frac{1}{2}, 1\right)$$

ratio of two independent solutions $\frac{u_1(x)}{u_2(x)}$

$u_j(x)$ - multivalued functions



$$\text{e.g. } \sqrt{x} \rightarrow -\sqrt{x}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = M_0 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad M \in \mathrm{GL}(2, \mathbb{Z})$$

γ -loop around singularity 0

M_0 - monodromy matrix

take another loop γ around another singularity, analyt. continue

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \xrightarrow{\gamma_*} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = M_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Monodromy of the fundamental group for linear equation is the representation $\pi_1(\mathbb{C} \setminus \text{singular points } x_0 \text{ of equation})$

$$\pi_1(\mathbb{C} \setminus \text{sing}) \rightarrow \mathrm{GL}(n, \mathbb{C})$$

↑
order of equation (the highest der.)

$$M_{P_1} M_{P_2} \dots M_{P_m} = I$$

simple example:

$$\text{let } u_1 = \sqrt{x}, u_2 = \sqrt{x-1}$$



$$\gamma_0 \left(\frac{\sqrt{x}}{\sqrt{x-1}} \right) = \begin{pmatrix} -\sqrt{x} & 1 \\ \sqrt{x-1} & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{x} \\ \sqrt{x-1} \end{pmatrix}$$

$$\gamma_1 \left(\frac{\sqrt{x}}{\sqrt{x-1}} \right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{x} \\ \sqrt{x-1} \end{pmatrix}$$

" M_1

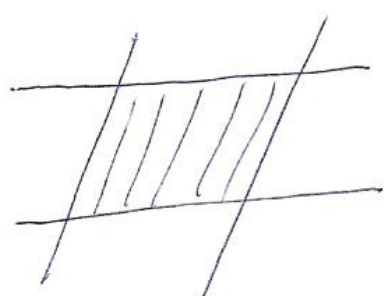
$$M_\infty = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = I_2$$

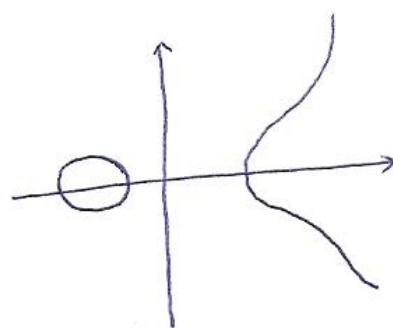
monodromy group of HGE $(\frac{1}{2}, \frac{1}{2}, 1)$ is $\Gamma(2) = \{g \in \text{SL}(2, \mathbb{Z}) \mid g = I_2 \pmod{2}\}$ (Ψ)

$$\mathbb{H} / \text{SL}(2, \mathbb{Z}) \simeq X(4) \quad \text{HGE } (\frac{1}{2}, \frac{1}{2}, 1)$$

$$\mathbb{H} / \Gamma(2) \xrightarrow{\sim} X(4) \quad \text{HGE } (\frac{1}{12}, \frac{5}{12}, 1)$$



$\mathbb{C} / \text{lattice} \xrightarrow{p(z)}$



cubic curve in $\mathbb{P}^2$

branched over 4 points

Θ fs

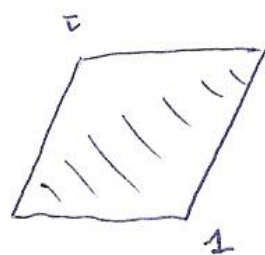
$\mathbb{P}^3$ intersection of quadratic's (Riemann's rel)

Θ - fs

line in $\mathbb{P}^2$
 $y_1 - y_2 + y_3 = 0$
Jacobi's identity

invariants

$X(4)$



isomorphism classes of elliptic curves

$\mathbb{H} / \text{SL}(2, \mathbb{Z})$ - not ordered by e_j $\xrightarrow{j\text{-invariant}} X(4)$

$\mathbb{H} / \Gamma(2)$ order d $\xrightarrow{[\lambda: 1: 1-\lambda]} X(4)$

$$y^2 = ax^3 + bx + c \subset \mathbb{P}^2 \rightsquigarrow P \quad \text{mero double-periodic}$$

$[x:y:1] \in \mathbb{P}^2$

Θ functions in $\mathbb{P}^3 \ni [x:y:z:w]$

$$\begin{cases} ax^2 + by^2 + cz^2 = 0 \\ ey^2 + fz^2 + gw^2 = 0 \end{cases}$$

$\rightsquigarrow \Theta$ - f - holo
1 period, quasi-period

$$X(4) \simeq \mathbb{C} / \langle 1, i \rangle, \quad X(4) \simeq \mathbb{C}$$

$$X(4) \cong \mathbb{C} \setminus \{0, 1\} \quad \tau \rightarrow \frac{e_1(\tau) - e_2(\tau)}{e_1(\tau) - e_3(\tau)} \quad - \lambda\text{-invariant}$$

$$X(43) \cong \mathbb{C} \quad \tau \rightarrow \frac{g_2^3}{g_2^3 - 27g_3^2} \quad - j\text{-invariant}$$

2 elliptic curves are isomorphic if either λ -invariant or j coincide

Periods of elliptic integrals

$$w^2 = G(z) \quad (*)$$

$$\int R(z, \sqrt{G(z)}) dz$$

L -closed path,

not through zeroes of G

and poles of R , on the Riemann's surface of $(*)$ (on torus)

$$I = \int_L R(z, \sqrt{4z^3 - g_2z - g_3}) dz = \int R(p(u), p'(u)) p'(u) du =$$

$$= \int \underbrace{F(u)}_{\text{elliptic function}} du$$

$$F(u) = c + \sum_{\substack{a \text{ - poles of} \\ F \text{ in } \mathbb{P}}} \{ A_0 \zeta(u-a) + A_1 p(u-a) + \dots + A_r p^{(r-1)}(u-a) \}, \quad \sum_a A_i = 0$$

$$p(u-a) = -\frac{d}{du} \zeta(u-a)$$

$$\zeta(u-a) = \frac{d}{du} \ln \sigma(u-a)$$

$$I = \int F(u) du = cu + \sum_a A \ln \sigma(u-a) - \sum_a A_1 \zeta(u-a) + \sum_a \underbrace{\{ A_2 p(u-a) + \dots + A_r p^{(r-2)}(u-a) \}}_{\substack{\text{elliptic function} \\ = \text{Rat}(p, p')}} =$$

fact $\zeta(u-a) - \zeta(u)$ - periodic ω_1, ω_2
 $\Rightarrow$ also elliptic

$$\sum_a A_i \zeta(u-a) = \zeta(u) \cdot \sum_a A_i + \sum_a A_i (\zeta(u-a) - \zeta(u)) = \text{const} \zeta(u) + R_2(p(u), p'(u))$$

$$\sum_a A_i = 0 \quad \sum_a A \ln \sigma(u-a) = \sum_a A_i (\ln \sigma(u-a) - \ln \sigma(u)) \stackrel{\text{identity of } \sigma}{=}$$

$$= \sum_{a \neq 0} A_i \ln \left\{ \frac{\sigma(u-a)}{\sigma(u)\sigma(a)} + \frac{\sigma'(a)}{\sigma(a)} u \right\} + R_3(p(u), p'(u))$$

$$I = \underbrace{cu}_{(1)} + \underbrace{c_2 \zeta(u)}_{(2)} + \sum_a A \ln \left\{ \frac{\sigma(a-u)}{\sigma(u)\sigma(a)} + \frac{\sigma'(u)}{\sigma(a)} u \right\} + R_4(p(u), p'(u))$$

$$p(u(z)) = z$$

$$p'(u(z)) = \sqrt{4z^3 - g_2z - g_3}$$

$$u = \int_L du = \int \frac{dz}{\sqrt{4z^3 - g_2z - g_3}} \quad \text{for (1)}$$

1st type elliptic integral

$$\zeta(u(z)) = - \int p(u) du = - \int \frac{p(u) dp(u)}{p'(u)} = - \int \frac{z dz}{\sqrt{4z^3 - g_2z - g_3}} \quad \text{for (2)}$$

2nd type elliptic integral

using

$$\frac{1}{2} \frac{p'(u) - p'(v)}{p(u) - p(v)} = \zeta(u+v) - \zeta(u) - \zeta(v)$$

$$\ln \left\{ \frac{\sigma(a-u)}{\sigma(a)\sigma(u)} + \frac{\sigma'(u)}{\sigma(a)} u \right\} = \ln \left\{ \frac{\sigma(a-u)}{\sigma(u)\sigma(a)} + \zeta(a)u \right\}$$

$$p(a) = z_0$$

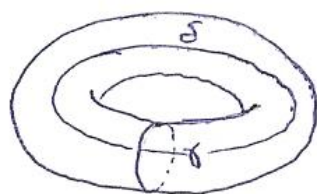
$$p'(a) = w_0$$

that (3) part in I is

$$\int \frac{1}{2} \frac{\sqrt{4z^3 - g_2z - g_3} + w_0}{z - z_0} dz$$

$$\Rightarrow I = \int R(z, \sqrt{\quad}) dz = \text{linear combination of integrals of 1st, 2nd, 3rd type} + \underbrace{R(z, \sqrt{4z^3 - g_2z - g_3})}_{\text{elliptic function}}$$

Periods:



$$J_1 = \int \frac{dz}{\sqrt{4z^3 - g_2z - g_3}} = m\omega_1 + m\omega_2$$

any closed loop not contractible to zero (eg. σ, δ)

$$\varphi(u) = R(p(u), p'(u)) p'(u)$$

$$\varphi(u) = 1$$

$$J_2 = \int_{\text{closed curve}} \frac{z dz}{w} = \int p(u) du = m\eta_1 + m\eta_2$$

$\varphi(u) = p(u)$

$$\int_{u_0}^{u_0 + w_1} p(u) du = -\zeta(u_0 + w_1) + \zeta(u_0) = -\eta_1$$

for 3rd type integral

$$\varphi(u) = \zeta(u-a) - \zeta(u) + \zeta(a)$$

$$\frac{1}{u-a} - \frac{1}{u}$$

$$\int \text{3rd type integral} = 2\pi i m + m_1 R_1 + m_2 R_2$$

27.05.2011

ODE's with elliptic functions weffs, Lamé equation

$$\frac{x^2}{a^2 + Q} + \frac{y^2}{b^2 + Q} + \frac{z^2}{c^2 + Q} = 1 \quad (1)$$

$-c^2 < Q \Rightarrow$ ellipsoid

$-b^2 < Q < -c^2$

$-a^2 < Q < -b^2$

λ, μ, ν - solutions of (1) (functions of x, y, z)

$$1 > -c^2 > \mu > -b^2 > \nu > -a^2$$

λ, μ, ν - new curvilinear coordinates

$$\Delta w = \sum \frac{\partial^2 w}{\partial x_i^2} = 0 \quad x_i(x, y, z)$$

change to λ, μ, ν

$$\leadsto \frac{\partial w}{\partial \lambda} \dots$$

$$\text{Take } \lambda = -(a \operatorname{cn} \alpha)^2 - (b \operatorname{sn} \alpha)^2$$

$$\mu = -(a \operatorname{cn} \beta)^2 - (b \operatorname{sn} \beta)^2$$

$$\nu = -(a \operatorname{cn} \gamma)^2 - (b \operatorname{sn} \gamma)^2$$

$$x = k^2(a^2 - c^2)^{1/2} \operatorname{sn} \alpha \operatorname{sn} \beta \operatorname{sn} \gamma$$

y, z } analogous

$$k^2 = \frac{a^2 - b^2}{a^2 - c^2}$$

$$W = A(\alpha) B(\beta) C(\gamma)$$

separation of variables

$$\leadsto \frac{A''}{A} = \lambda (\operatorname{sn} \alpha)^2 - h$$

$$\frac{B''}{B} = \lambda (\operatorname{sn} \beta)^2 - h \quad \lambda, h - \text{const.}$$

$$\frac{C''}{C} = \lambda (\operatorname{sn} \gamma)^2 - h$$

Lamé equation in Jacobi form

$$\frac{d^2 \lambda(z)}{dz^2} + \{h - n(n+1)\} [k \operatorname{sn}(z, k)^2] \lambda(z) = 0$$

many "PDEs" can be reduced to $\uparrow$ by changing to some "exotic" coordinates.

Lame equation with Weierstrass function:

$$\frac{d^2 \lambda}{du^2} + [H - n(n+1)p(u)] \lambda = 0$$

algebraic form of Lame equation

$$(\operatorname{sn} x)^2 = x$$

$$(\operatorname{sn}'(u))^2 = (1 - \operatorname{sn}^2(u))(1 - k^2 \operatorname{sn}^2 u)$$

$$\frac{d^2 \lambda}{dx^2} + \frac{1}{2} \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-k^2} \right) \frac{d\lambda}{dx} + \frac{hk^2 - n(n+1)x}{4x(x-1)(x-k^2)} \lambda = 0$$

"Fuchsian type"

singular points $0, 1, k^{-2}, \infty$

in neighbourhoods of all these singularities $\exists$ 2 independent

solutions $\lambda = (x-a)^r \sum_{n=0}^{\infty} a_n (x-a)^n, r \in \mathbb{Q}$

or $1/x$

Riemann's scheme

$$\left\{ \begin{array}{cccc} 0 & 1 & k^{-2} & \infty \\ \hline 0 & 0 & 0 & -\frac{n}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{n+1}{2} \end{array} \right\}$$

$\uparrow$

$$y = (x-1)^0 \sum a_n (x-1)^n$$

$$y = (x-1)^{1/2} \sum b_n (x-1)^n$$

another change of variables:

$$p = p(u)$$

$$\frac{d^2 \lambda}{dp^2} + \frac{1}{2} \left(\frac{1}{p-e_1} + \frac{1}{p-e_2} + \frac{1}{p-e_3} \right) \frac{d\lambda}{dp} + \frac{H - (n+1)np}{4(p-e_1)(p-e_2)(p-e_3)} = 0$$

These forms are particular cases Fuchs's equation:

$$\frac{dw}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a} \right) \frac{dw}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-a)} w$$

$$\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0$$

$\exists$ transforms

R. Burger, G. Labahn, M. van Hoeig

"Closed form solutions of linear ODEs having elliptic functions coefficients"

$$a_n(x)y^{(n)} + \dots + a_1(x)y'(x) + a_0(x)y(x) = 0 \quad (*)$$

a_i - doubly-periodic, meromorphic functions (elliptic) with same periods, for instance: p, σ , ratios of theta functions

Lame eq. $n=2$.

Theorem (Picard)

If solutions are uniform (path independent) $a_i \not\equiv 0$

$\Rightarrow$ ODE (*) has solutions which are doubly periodic

of second kind (which are expressed by ξ, σ -f)

Def: $F(x)$ is doubly-periodic of second kind if

$$\exists T, T' \in \mathbb{C}, \quad \exists \delta, \delta' \in \mathbb{C} \quad \text{s.t.} \quad F(x+T) = \delta F(x), \quad F(x+T') = \delta' F(x)$$

$$\forall x \in \mathbb{C}.$$

Remark for (*): one can express (*) in this form:

$$\mathbb{C}(x, \sqrt{w(x)}) \left[\frac{d}{dx} \right]$$

(linear ODE with rational coefficients in $x, \sqrt{w(x)}$ where

$w \in \mathbb{C}[x]$ polynomial)

$$x \rightarrow f$$

$$\sqrt{w(x)} \rightarrow f'$$

$$(f')^2 = w(t)$$

after this change of variables we get

$$\mathbb{C}(f, f') \left[\frac{d}{df} \right]$$

Theorem (Picard): was already $\checkmark$

Fact: $F(x)$ - 2nd kind $\Leftrightarrow G(x) = \frac{F'(x)}{F(x)}$ - doubly periodic

$$\text{Proof: } \Rightarrow G(x+T) = \frac{F'(x+T)}{F(x+T)} = \frac{\delta F'(x)}{\delta F(x)} = G(x)$$

$$\Leftarrow \frac{d \log \left(\frac{F(x+T)}{F(x)} \right)}{dx} = G(x+T) - G(x) = 0$$

$$\Rightarrow \exists c: F(x+T) = c F(x).$$

(*) - coming back

$\{f_1(x), \dots, f_n(x)\}$ - basis of solutions

$f_i(x+T)$ - solutions (*)

$\{f_i(x+T)\}$ - other basis

$$f_i(x+T) = a_{i1}f_1(x) + \dots + a_{in}f_n(x), \quad \vec{f}(x+T) = \vec{f}(x) \cdot A$$

$$\vec{f}(x+T') = \vec{f}(x) \cdot B$$

Theorem: A, B - constant matrices associated to a given basis of solutions of (*) $\{f_1(x), \dots, f_n(x)\}$

$\Rightarrow$ 1) A, B - nonsingular

2) $\det A = \frac{W(\bar{F}(x+T'))}{W(\bar{F}(x))}$ W - Wronskian ; $\det B = \dots$

3) Products of eigenvalues of A, B are

$$\exp \left(- \int_x^{x+T} \frac{a_{n-1}(u)}{a_n(u)} du \right)$$

$$= 1 - T'$$

4) if $F(x)$ is a solution $F(x) = c_1 f_1 + \dots + c_n f_n \Rightarrow$

$F(x)$ - doubly periodic 2nd kind with multipliers $s, s' \Leftrightarrow$

s, s' are eigenvalues of A, B

$\bar{c} = (c_1 \dots c_n)^T$ - common eigenvector of A, B ($AB=BA$)

Remark 1) Picard's theorem follows from 4)

2) once 1 solution of 2nd kind is found

$\Rightarrow$ can use reduction of order of ODE to find other solutions of 2nd kind.

Theorem: Suppose $a_{n-1}(x) = 0$ and $y_1, \dots, y_n$ - basis of solutions

doubly periodic of second kind.

$\Rightarrow z(x) = y_1 \dots y_n(x)$ - doubly periodic.

Proof: $\Rightarrow$ from prev thm 3) $s_1, \dots, s_n = 1$

$$y_i(x+T) = s_i y_i(x)$$

$$y_i(x+T') = s_i' y_i(x)$$

$$z(x+T) = s_1 \dots s_n y_1 \dots y_n = z(x)$$

$n=2$

$$y''(x) - r(x)y(x) = 0 \quad (1) \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$y'''(x) - 4r(x)y'(x) - 2r'(x)y(x) = 0 \quad (2) \quad (y_1^2, y_1 y_2, y_2^2)$$

if (1) has a basis of solutions of 2nd kind $\Rightarrow$

(2) has at least 1 solution $z(x)$ which is doubly periodic

Example: Lamé equation (2): $y'''(x) - 4(n(n+1)p(x) + \beta)y' - 2n(n+1)p'(x)y = 0$

we can find elliptic solutions

$$\text{if } n=1 \quad y_1 = \frac{\sigma(x+a)}{\sigma(x)\sigma(a)} e^{-x\zeta(a)}$$

$$y_2 = \frac{\sigma(x-a)}{\sigma(x)\sigma(a)} e^{x\zeta(a)}$$