Wykład Karcha

Równanie ciepła na $\mathbb{R}^n$

$$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = u_0(x) \end{cases} \qquad u = u(x,t) \text{ - niewiadoma}$$

Tw (Klasyczne, zob. Evans)

Dla danego warunku początkowego $u_0 \in L^1(\mathbb{R}^n)$ funkcja

$$u(x,t) = \int_{\mathbb{R}^n} G(x-y,t)\,u_0(y)\,dy$$

gdzie $G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ jądro ciepła, jądro Gaussa-Weierstraßa

jest rozwiązaniem tego zagadnienia

Dodatkowo $u = u(x,t)$ ma własności:

1° $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$

2° $\| u(\cdot,t) - u_0(\cdot)\|_1 \to 0, \quad t \to 0$

3° $\|u(\cdot,t)\|_1 \le \|u_0\|_1$

4° $\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_0(x)\,dx$

To jest jedyne rozwiązanie o tych własnościach

Dowód 3°:

$$\int_{\mathbb{R}^n} |u(x,t)|\,dx \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G(x-y,t)\,u_0(y)\,dy \overset{\text{Fubini} + \int_{\mathbb{R}^n} G(y,t)dy = 1}{\le} \int_{\mathbb{R}^n} |u_0(y)|\,dy$$

Dowód 4°: Podobny tylko Fubini

Wniosek:

$$|G(x,t)| = |(4\pi t)^{-n/2} e^{-|x|^2/4t}| \le (4\pi t)^{-n/2}$$

~~Wniosek 2:~~ zatem $|u(x,t)| \le (4\pi t)^{-n/2} \int_{\mathbb{R}^n} |u_0(y)|\,dy$

Zatem $u(x,t) \sim t^{-n/2}$, gdzie $t \to \infty$

Wniosek 2: Dla $p \in [1,\infty]$

$$\|u(\cdot,t)\|_p = \left(\int_{\mathbb{R}^n} |u(x,t)|^p dx\right)^{1/p} \le \left(\|u(\cdot,t)\|_\infty^{p-1} \int_{\mathbb{R}^n} |u(x,t)|\,dx\right)^{1/p} \le$$

$$\le (4\pi)^{-n/2 \, \frac{p-1}{p}}\, t^{-\frac{n}{2}\left(\frac{p-1}{p}\right)} \|u_0\|_1^{\frac{p-1}{p}+\frac{1}{p}} = C(p,n)\cdot t^{-n/2\left(1-\frac{1}{p}\right)} \|u_0\|_1$$

Wniosek 3:

$\forall\ p \in [1,\infty]$

$$\|\nabla u(\cdot,t)\|_p \le C t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}} \|u_0\|_1$$

dowód:

$$\nabla_x u(x,t) = \int_{\mathbb{R}^n} \nabla_z G(\underbrace{x-y}_{z}, t)\,u_0(y)\,dy$$

Prosty rachunek

$$\partial_{x_k} G(x,t) = \partial_{x_k}\left[(4\pi t)^{-n/2} e^{-\left|\frac{x}{2\sqrt{t}}\right|^2}\right] = (4\pi t)^{-n/2} \frac{1}{2\sqrt{t}} \underbrace{\left(-\frac{2x_k}{2\sqrt{t}}\right) e^{-\left|\frac{x}{2\sqrt{t}}\right|^2}}_{\text{ogr}}$$

A dalej jak we wniosku 2

Tw (o asymptotyce)

Dla kazdego $u_0 \in L^1(\mathbb{R}^n)$ i kazdego $p \in [1, \infty]$ rozwiązanie z tw. klasycznego spełnia

$$t^{\frac{n}{2}(1-\frac{1}{p})} \|u(\cdot,t) - MG(\cdot,t)\|_p \xrightarrow{t\to\infty} 0, \quad \text{gdzie} \quad M = \int_{\mathbb{R}^n} u_0(y)dy$$

Uwaga:

Zauwazmy, ze

$$G(x,t) = (4\pi t)^{-n/2} e^{-|\frac{x}{2\sqrt{t}}|^2} = t^{-n/2}\, G(\tfrac{x}{\sqrt{t}}, 1)$$

$$\|G(\cdot,t)\|_p = t^{-n/2} \|G(\tfrac{\cdot}{\sqrt{t}}, 1)\|_p = t^{-n/2}\Big(\int_{\mathbb{R}^n} |G(\tfrac{x}{\sqrt{t}}, 1)|^p dx\Big)^{1/p} \overset{\substack{x = \sqrt{t}y \\ dx = t^{n/2}dy}}{=}$$

$$= t^{-n/2 + n/2p} \|G(\cdot,1)\|_p = t^{-n/2(1-\frac{1}{p})} \|G(\cdot,1)\|_p$$

Dowód tw:

KROK 1: $u_0 \in C_0^\infty(\mathbb{R}^n)$

tw. o wartości średniej

kazdą funkcję $h \in C^1(\mathbb{R}^n)$ mozna zapisać jako

$$h(x) - h(x-y) = \int_0^1 \langle (\nabla h)(x-(1-s)y), y \rangle ds$$

dowód: $F(s) = h(x-y+sy)$

wtedy $h(x) - h(x-y) = F(1) - F(s) = \int_0^1 F'(s)ds$

szacujemy różnicę

$$|u(x,t) - MG(x,\overset{t}{y})| = |\int_{\mathbb{R}^n} u_0(y)[G(x-y,t) - G(x,t)]dy| \le ct^{-n/2-1/2} \int_{\mathbb{R}^n} |u_0(y)||y|dy$$

zatem

$$t^{n/2} \|u(\cdot,t) - MG(\cdot,t)\|_\infty \le ct^{-1/2} \int_{\mathbb{R}^n} |u_0(y)||y|dy$$

$$\|u(\cdot,t) - MG(\cdot,t)\|_1 \le ct^{-1/2} \int_{\mathbb{R}^n} |u_0(y)||y|dy$$

podobnie

$$\|u(\cdot,t) - MG(\cdot,t)\|_1 \le ct^{-1/2} \int_{\mathbb{R}^n} |u_0(y)||y|dy$$

zatem

$$t^{\frac{n}{2}(1-\frac{1}{p})} \|u(\cdot,t) - MG(\cdot,t)\|_p \le t^{n/2(1-\frac{1}{p})} \|u(\cdot,t) - MG(\cdot,t)\|_\infty^{\frac{p-1}{p}} \|u(\cdot,t) - MG(\cdot,t)\|_1^{1/p} \le$$

$$\le ct^{-1/2} \int_{\mathbb{R}^n} |u_0(y)||y|dy$$

Krok 2: Niech $u_0 \in L^1(\mathbb{R}^n)$

$\varphi \in C_0^\infty(\mathbb{R}^n)$ $\quad \int_{\mathbb{R}^n} \varphi(x)dx = \int_{\mathbb{R}^n} u_0(x)dx, \quad \|\varphi - u_0\|_1 \le \varepsilon$

Szacujemy:

$$t^{\frac{n}{2}(1-\frac{1}{p})} \|u(\cdot,t) - MG(\cdot,t)\|_p \le t^{n/2(1-\frac{1}{p})} [\|G(\cdot,t) * u_0 - G(\cdot,t) * \varphi\|_p + \|G(\cdot,t) * \varphi - MG(\cdot,t)\|_p]$$

$$\le C\|u_0 - \varphi\|_1 + \underbrace{t^{n/2(1-\frac{1}{p})} \|G(\cdot,t) * \varphi - MG(\cdot,t)\|_p}_{(*)} \le C\varepsilon + \underset{0}{\overset{(*)}{\downarrow t\to\infty}} \quad \text{z kroku 1}$$

Lemat (uwaga do tw. o asymptotyce)

są równoważne

- $t^{\frac{n}{2}(1-\frac{1}{p})} \| u(\cdot,t) - MG(\cdot,t)\|_p \xrightarrow{t\to\infty} 0$
- istnieje $t_0 > 0$ t. że $\lambda^n u(\lambda x, \lambda^2 t_0) \xrightarrow[w\ L^p(\mathbb{R}^n)]{\lambda\to\infty} MG(x,t_0)$

Dowód:

$G(x,t_0) = t_0^{-n/2} G(\frac{x}{\sqrt{t_0}}, 1)$

$\| \lambda^n u(\lambda\cdot, \lambda^2 t_0) - MG(\cdot,t_0)\|_p = \| \lambda^n u(\lambda\cdot, \lambda^2 t_0) - M\lambda^n G(\lambda\cdot, \lambda^2 t_0)\|_p = \lambda^{n-\frac{n}{p}} \|u(\cdot,\lambda^2 t_0) - MG(\cdot,\lambda^2 t_0)\|_p$

$\underset{\lambda=\sqrt{\frac{t}{t_0}}}{=} \left(\frac{t}{t_0}\right)^{\frac{n}{2}(1-\frac{1}{p})} \|u(\cdot,t) - MG(\cdot,t)\|_p$ stąd równoważność

szkic innego dowodu tw. o asymptotyce

Metoda 4 kroków

Krok 1: skalowanie

definiujemy $u_\lambda(x,t) = \lambda^n u(\lambda x, \lambda^2 t)$

Krok 2: Oszacowania i zwartość

Pokazujemy, że ~~istnieje~~ $\forall\ t_0 > 0$ takie że rodzina $\{u_\lambda(\cdot,t_0)\}_{\lambda>0}$ istnieje zwarta w $L^p(\mathbb{R}^n$

Krok 3: Przejście graniczne

istnieje podciąg zbieżny

$u_{\lambda_n}(\cdot,t_0) \xrightarrow[w\ L^p(\mathbb{R}^n)]{\lambda_n\to\infty} \bar{u}(\cdot,t_0)$

łatwo pokazać $\bar{u}_t = \Delta\bar{u}$

Krok 4: Identyfikacja granicy

warunek początkowy

$\lambda_k^n u_0(\lambda_k x) \xrightarrow[\text{w sensie miarowym}]{\lambda_k\to\infty} M\delta_0$

czyli

$$\begin{cases} \bar{u}_t = \Delta\bar{u} \\ \bar{u}(x,0) = M\delta_0 \end{cases}$$

Jedynym rozwiązaniem jest $\bar{u}(x,t) = MG(x,t)$

$u_\lambda(x,t) \xrightarrow[w\ L^p(\mathbb{R}^n)]{\lambda\to\infty} MG(x,t)$

18.03.2011

Równanie ciepła

$u_t = \Delta u$ $x\in\mathbb{R}^n$, $t>0$

szczególne rozwiązanie

$G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$

ważna własność:

$\forall\ \lambda>0$ $\lambda^n G(\lambda x, \lambda^2 t) = G(x,t)$

Def: Rozwiązanie nazywa się samopodobne (self - similar) jeżeli jest ono niezmiennicze na pewne skalowania.

Uwaga: Równanie ciepła jest niezmiennicze na skalowanie $u_\lambda(x,t) = \lambda^n u(\lambda x, \lambda^2 t)$
Jeżeli u jest, to u_λ też jest rozwiązaniem.

Uwaga: G nie jest jedynym rozwiązaniem samopodobnym równania ciepła, inne rozwiązanie $\partial_{x_k} G$, pochodne wyższych rzędów

Lemat: $\forall\, u_0 \in L^1(\mathbb{R}^n)$ $\forall\, \varphi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \lambda^n u_0(\lambda x)\varphi(x)\,dx \xrightarrow{\lambda \to \infty} M\varphi(0) \quad , \quad \text{gdzie} \quad M = \int_{\mathbb{R}^n} u_0(x)\,dx$$

(t.zn. $\lambda^n u_0(\overset{\lambda}{x}) \xrightarrow[\text{słabo, w sensie miar}]{} M\delta_0$)

Dowód:
Zamiana zmiennych

$$\int_{\mathbb{R}^n} \lambda^n u_0(\lambda x)\varphi(x)\,dx = \left[x = \frac{y}{\lambda},\ dx = \lambda^{-n}dy\right] = \int_{\mathbb{R}^n} u_0(y)\varphi\left(\frac{y}{x}\right)dy$$

tw. Leb. o zb. zmaj.

Słabe rozwiązania równania ciepła

Niech $u = u(x,t)$ będzie regularne

$$\begin{cases} u_t = \Delta u \quad /\varphi \\ u(x,0) = u_0(x) \end{cases} \qquad \begin{aligned} &\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty)) \\ &\varphi = \varphi(x,t) \end{aligned}$$

Całkujemy po $\mathbb{R}^n \times [0,\infty)$:

$$0 = \int_0^\infty \int_{\mathbb{R}^n} (u_t - \Delta u)\varphi\,dx\,dt \overset{\text{przez części}}{=} \underbrace{-\int_{\mathbb{R}^n} u(x,0)\varphi(x,0)\,dx - \int_0^\infty \int_{\mathbb{R}^n} (\varphi_t + \Delta\varphi)u\,dx\,dt}_{(*)}$$

Def: Mówimy, że funkcja $u \in L^1_{loc}(\mathbb{R}^n \times [0,\infty))$ jest słabym rozwiązaniem równania $u_t - \Delta u = 0$, $u(x,0) = u_0(x)$, jeżeli

$0 = (*)$ dla wszystkich $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$

Def:2 $u \in L^1_{loc}(\mathbb{R}^n \times [0,\infty))$ jest słabym rozwiązaniem zagadnienia

$$\begin{cases} u_t = \Delta u \\ u(x,0) = M\delta_0 \end{cases}$$

jeżeli

$$0 = -M\varphi(0,0) - \int_0^\infty \int_{\mathbb{R}^n} (\varphi_t + \Delta\varphi)u\,dx\,dt \quad \text{dla wszystkich} \quad \varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$$

Przykład:
$G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ spełnia definicję 2 z $M = 1$

Niech $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$, $\varepsilon > 0$

$$-\int_\varepsilon^\infty \int_{\mathbb{R}^n} (\varphi_t + \Delta\varphi)G\,dx\,dt \overset{\text{p. części}}{=} + \int_{\mathbb{R}^n} \varphi(x,\varepsilon)G(x,\varepsilon)\,dx + \int_\varepsilon^\infty \int_{\mathbb{R}^n} \varphi\overset{=0}{(G_t - \Delta G)}\,dx\,dt =$$

$$= \int_{\mathbb{R}^n} \varphi(x, \varepsilon)\, \varepsilon^{-n/2}\, G\left(\frac{x}{\varepsilon^{1/2}}, 1\right) dx \quad = \text{zam. zmiennych} \longrightarrow \varphi(0,0)$$

Twierdzenie (o jednoznaczności)

Załóżmy, że $v \in C(\mathbb{R}^n \times (0,\infty))$ spełnia $\sup_{t>0} \|v(\cdot,t)\|_1 < \infty$ jest słabym rozwiązaniem zagadnienia

$$V_t = \Delta v \qquad x \in \mathbb{R}^n$$
$$v(x,0) = M\,\delta_0$$

Wówczas $v(x,t) = M\,G(x,t)$

Dowód: Załóżmy, że mamy 2 takie rozwiązania v_1, v_2.
wtedy (z liniowości) $w = v_1 - v_2$ jest słabym rozwiązaniem

$$\begin{cases} w_t = \Delta w \\ w(x,0) = 0 \end{cases}$$

t.zn. że $(*)$ $\displaystyle\int_0^\infty \int_{\mathbb{R}^n} (\varphi_t + \Delta\varphi) w \,dx dt = 0 \qquad \forall\ \varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$

Mamy pokazać, że $w \equiv 0$

własności $w = w(x,t) = v_1 - v_2$

(i) $w \in C(\mathbb{R}^n \times (0,\infty))$

(ii) $\sup_{t \geq 0} \|w(\cdot,t)\|_1 < \infty$

(iii) $(*)$ $\displaystyle\int_0^\infty \int_{\mathbb{R}^n} (\varphi_t + \Delta\varphi) w \,dx dt = 0 \qquad \forall\ \varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$

Pokażemy, że $(*)$ jest prawdziwa dla dowolnej funkcji $\varphi \in C^\infty(\mathbb{R}^n \times [0,\infty))$ spełniającej:

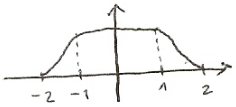
a) $\sup_{t>0} \|\partial_t \varphi(\cdot,t)\|_\infty < \infty$

b) $\sup_{t>0} \|\Delta\varphi(\cdot,t)\|_\infty < \infty$

c) $\operatorname{supp} \varphi \subseteq \mathbb{R}^n \times [0,T)$ dla pewnego $T > 0$

Niech $\Theta(s) \in C_0^\infty(\mathbb{R}^n)$

$\Theta_j(x) = \Theta\left(\frac{|x|}{j}\right)$



Niech φ spełnia (a) - (c) wtedy $\varphi_j = \varphi\Theta_j \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$

wstawiamy do $(*)$

czyli $\displaystyle\int_0^\infty \int_{\mathbb{R}^n} (\partial_t \varphi_j + \Delta\varphi_j) w \,dx dy = 0$

$$j \to \infty$$

tw. Lebesgue'a o zb. zmajoryzowanej

$$\int_0^\infty \int_{\mathbb{R}^n} (\varphi_t + \Delta\varphi) w \,dx dt = 0$$

Krok 2: Niech $\psi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$ t.że $\operatorname{supp}\psi \subset \mathbb{R}^n \times [0,T)$
wówczas istnieje $\varphi \in C^\infty(\mathbb{R}^n \times [0,\infty))$ o własnościach (a) - (c) t.że

$$\varphi_t + \Delta\varphi = \psi$$

Konstrukja φ

Rozwiązanie zagadnienia

$$\begin{cases} \varphi_t + \Delta\varphi = \psi & \text{dla} \quad t < T \ , \ x \in \mathbb{R}^n \\ \varphi(x,T) = 0 \end{cases}$$

zamiana zmiennych $s = T - t$

otrzymujemy problem
$$\begin{cases} -\varphi_t + \Delta\varphi = \psi & x \in \mathbb{R}, t > 0 \\ \varphi(x,0) = 0 \end{cases}$$
jawne rozwiązanie z Evansa

Pokazaliśmy, że

$$\int_0^\infty \int_{\mathbb{R}^n} \psi w \, dx dt = 0 \qquad \forall \psi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$$

$$\Downarrow$$

$$w \equiv 0$$

Asymptotyka rozwiązań równania konwekcji - dyfuzji

$$\begin{cases} u_t - u_{xx} + (u^q)_x = 0 \ , & x \in \mathbb{R} \ , t > 0 \\ u(x,0) = u_0(x) \end{cases}$$
(Przypadek 1 wymiarowy)

stałe $q > 1$

Tw. (o istnieniu)

Założenia $u_0 \in L^1(\mathbb{R})$, $u_0 \geq 0$, $q > 1$ - parametr. Przy tych założeniach zagadnienie (KD) ma $\exists$ dokładnie jedno rozwiązanie o własnościach

1° $u \in C^{2,1}(\mathbb{R} \times (0,\infty)) \cap L^\infty([0,\infty), L^1(\mathbb{R}))$

2° $\|u(\cdot,t) - u_0(\cdot)\|_1 \to 0 \ , \ t \to 0$

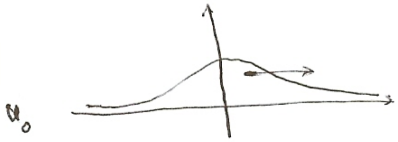
3° $u(x,t) \geq 0$ i wiele innych własności

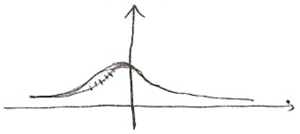
np. $u(x,t) \to 0 \ , \ |x| \to \infty$

Motywacje:

$u_t + (u^q)_x = 0$

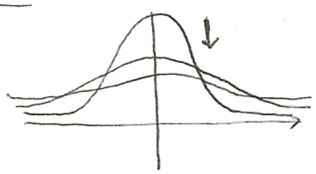
metoda charakterystyk





zalamują się
i maleją
szybciej
się załamują

Dyfuzja



Metoda 4 kroków (J.L. Vazquéz)

Krok 1: Skalowanie

$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ (bo chcemy zachować dyfuzję)

$u(x,t) = \frac{1}{\lambda} u_\lambda(\frac{x}{\lambda}, \frac{t}{\lambda^2})$ wstawiamy do równania

$$\lambda^{-1-2} \partial_t u_\lambda - \lambda^{-1-2} \partial_x^2 u_\lambda + \lambda^{-q-1} (u_\lambda^q)_x = 0$$

$$\partial_t u_\lambda - \partial_x^2 u_\lambda + \lambda^{-q+2} (u_\lambda^q)_x = 0$$

3 przypadki:

$q > 2$ - asymptotyka liniowa

$q = 2$ równowaga

$1 < q < 2$ - asymptotyka nieliniowa

Zadanie na dzisiaj

Pokazać, że

$$\lambda u(\lambda x, \lambda^2 t) \xrightarrow{\lambda \to \infty} M\,G(x,t) \qquad \text{dla} \quad q > 2, \quad M = \int u_0$$

Krok 2: Oszacowania i zwartość

$$\begin{cases} u_t - u_{xx} + (u^q)_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Lemat 1:

$$\forall\, t>0 \qquad \int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_0(x)\,dx$$

Dowód:

$$\frac{d}{dt}\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_t(x,t)\,dx \overset{\text{z równania}}{=} \int_{\mathbb{R}} u_{xx} - (u^q)_x\,dx = u_x(x,t)\Big|_{x=-\infty}^{x=+\infty} - u^q(x,t)\Big|_{-\infty}^{\infty} = 0$$

Powyższy argument został nieprzyjęty przez słuchaczy

~~Jeżeli u jest~~ (ten też :))

Niejednorodne równanie ciepła

$$\begin{cases} u_t - u_{xx} = f(x,t) \\ u(x,0) = u_0(x) \end{cases} \qquad \begin{aligned} &x \in \mathbb{R},\ t>0 \\ &f = f(x,t) \text{ - dana np } f \in C(\mathbb{R} \times (0,\infty)) \end{aligned}$$

Tw: Rozwiązanie dane jest wzorem

$$u(x,t) = \underbrace{G(\cdot,t) * u_0}_{\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = u_0 \end{cases}} + \underbrace{\int_0^t \int_{\mathbb{R}} G(x-y,t-s) f(y,s)\,dy\,ds}_{\begin{cases} u_t - u_{xx} = f \\ u(x,0) = 0 \end{cases}} \qquad \text{Evans (str. 70)}$$

przeformułowanie (KD). Jeżeli istnieje rozwiązanie, to wyrażone jest wzorem

$$u(x,t) = \int_{\mathbb{R}} G(x-y,t) u_0(y)\,dy - \int_0^t \int_{\mathbb{R}} G(x-y,t-s)\,(u^q(y,s))_y\,dy\,ds \qquad \text{(wzór Duhamela)}$$

Stosujemy tw. Banacha do wzoru (D) i mamy istnienie w jakiś przestrzeniach Banacha

$$X = \left\{ u \in C^{2,1}(\mathbb{R} \times (0,T)) : \forall\, t>0 \ \begin{aligned} &u(x,t) \to 0, \ |x| \to \infty \\ &u_x(x,t) \to 0, \ |x| \to \infty \\ &u_{xx}(x,t) \to 0, \ |x| \to \infty \end{aligned} \right\}$$

$$\int_{\mathbb{R}} u_{xx}(x,t)\,dx = \lim_{R \to \infty} u_x(x,t)\Big|_{x=-R}^{x=R} = 0$$

Lemat 1:

$$M = \int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_0(x)\,dx$$

$$\|u(t)\|_1 = \|u_0\|_1$$

Lemat 2:

$\forall\ p \in [1,\infty]$ $\quad \exists\ C = C(p) > 0$

t. że

$$\|u(\cdot,t)\|_p \leq C t^{-\frac{1}{2}(1-\frac{1}{p})} \|u_0\|_1$$

Dowód: $p=1$ zrobione, $p=2$:

$$\frac{d}{dt}\int_{\mathbb{R}} u^2(x,t) = 2\int_{\mathbb{R}} uu_t\,dx \overset{\text{z r-nia}}{=} 2\int_{\mathbb{R}} uu_{xx}\,dx - 2\int_{\mathbb{R}} u(u^q)_x\,dx = -2\int_{\mathbb{R}}(u_x)^2\,dx$$

$$\underbrace{-2\frac{q}{q-1}\int_{\mathbb{R}}(u^{q+1})_x\,dx}_{\overset{\shortparallel}{0}}$$

$$\frac{d}{dt}\int_{\mathbb{R}} u^2\,dx = -2\int_{\mathbb{R}}(u_x)^2\,dx$$

Nierówność Nasha (pan z „Pięknego umysłu") (Sobolewa - Gagliarda - Nirenberga)

$\exists\ C_N > 0 \quad \forall u \in L^1(\mathbb{R}) \quad u_x \in L^2(\mathbb{R})$

$$\|u\|_2^6 \leq C_N \|u_x\|_2^2 \|u\|_1^4 .$$

$$\frac{d}{dt}\|u(t)\|_2^2 = -2\|u_x\|_2^2 \leq -\frac{2}{C_N\|u_0\|_1^4}\|u\|_2^6$$

$$f(t) = \|u(t)\|_2^2$$

$$f' \leq -C_1 f^3 \implies f(t) \leq C_2 t^{-1/2}$$

Wniosek:

$\forall\ p \in [1,\infty]$ $\quad \exists\ C(p) > 0$

$$\|u_\lambda(\cdot,t)\|_p \leq \underbrace{C t^{-\frac{1}{2}(1-\frac{1}{p})}\|u_0\|_1}_{\text{tu nie ma } \lambda} \qquad \forall \lambda > 0$$

Dowód:

$$\|u_\lambda(\cdot,t)\|_p = \|\lambda u(\lambda\cdot,\lambda^2 t)\|_p = \lambda^{1-\frac{1}{p}}\|u(\cdot,\lambda^2 t)\|_p \leq \lambda^{1-\frac{1}{p}} C(\lambda^2 t)^{-\frac{1}{2}(1-\frac{1}{p})}\|u_0\|_1 =$$

$$= C t^{-\frac{1}{2}(1-\frac{1}{p})}\|u_0\|_1$$

Lemat 3: $\forall\ p \in [1,\infty]$ $\quad \exists\ C = C(p) > 0$

$$\|\partial_x u(\cdot,t)\|_p \leq C t^{-1/2(1-\frac{1}{p})-\frac{1}{2}}\|u_0\|_1$$

Szkic dowodu:

$$u_t - u_{xx} + (u^q)_x = 0 \qquad w = u_x$$

$$w_t - w_{xx} + (wqu^{q-1})_x = 0$$

Wniosek: Po lematu można dostawić λ.

Tw (o zwartości) (Aubin - Lions - Arzela - Ascoli)

Załóżmy, że przestrzenie Banacha spełniają

$$X \underset{\text{zwarte}}{\subset} B \underset{\text{ciągłe}}{\subset} Y$$

Zakładamy, że zbiór F ma własności

1° F jest ograniczony w $C([0,T], X)$

2° $\{f_t : f \in F\}$ ograniczony w $C([0,T], Y)$

& wówczas

F jest warunkowo zwarty w $C([0,T], B)$

7.04.2011

Convection - diffusion equation:

$$\begin{cases} u_t - u_{xx} + (u^q)_x = 0 & x \in \mathbb{R},\ t>0 \\ u(x,0) = u_0(x) \end{cases}$$

Remark: (about heat equation, Evans 2.3)
Assume that $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then the function

$$u(x,t) = \int_{\mathbb{R}} G(x-y, t)\, u_0(y)\, dy$$

is a solution of heat equation with properties

a) $u \in C^\infty(\mathbb{R} \times (0,\infty))$

b) $\|u(\cdot,t)\|_\infty \le \|u_0\|_\infty$, $\|u(\cdot,t)\|_1 \le \|u_0\|_1$

c) $\|u(\cdot,t) - u_0(\cdot)\|_\infty \xrightarrow{t\to 0} 0$ and $\|u(\cdot,t) - u_0(\cdot)\|_1 \xrightarrow{t\to 0} 0$

There is unique solution with these properties

Convection equation:

$$\begin{cases} u_t + (u^q)_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

By the method of characteristics solution of this problem may not exist for all $t>0$

Nonhomogenious heat equation:

$$\begin{cases} u_t - u_{xx} = f(x,t) \\ u(x,0) = 0 \end{cases} \qquad f \in C^{2,1}(\mathbb{R} \times [0,\infty)) \qquad (N-H)$$

Theorem (Evans): DUHAMEL PRINCIPLE

The function $u(x,t) = \int_0^t \int_{\mathbb{R}} G(x-y, t-s) f(y,s)\, dy\, ds$ is a solution of problem (N-H) with the following properties

a) $u \in C^{2,1}(\mathbb{R} \times (0,\infty))$

b) $\|u(\cdot,t)\|_\infty \xrightarrow{t\to 0} 0$ and $\|u(\cdot,t)\|_1 \xrightarrow{t\to 0} 0$

This is a unique solution with these properties

Corollary:

The solution of the problem $\begin{cases} u_t - u_{xx} = f(x,t) \\ u(x,0) = u_0(x) \end{cases}$

has the form

$$u(x,t) = \int_{\mathbb{R}} G(x-y,t)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}} G(x-y,t-s)f(y,s)\,dy\,ds$$

Existence of solution to the C-D equation:

$$\begin{cases} u_t - u_{xx} + \underset{\overset{\shortparallel}{f}}{(u^q)_x} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

"equivalent" formula

$$u(x,t) = \int_{\mathbb{R}} G(x-y,t)u_0(y)\,dy - \int_0^t \int_{\mathbb{R}} G(x-y,t-s)(u^q)_y(y,s)\,dy\,ds$$

we'll show that this equation has a solution

$$u = \mathcal{F}(u)$$

we introduce the Banach space

$$X_T = C([0,T], L^\infty(\mathbb{R}) \cap C(\mathbb{R}))$$

with the norm

$$\|u\|_{X_T} = \sup_{0 \le t \le T} \|u(\cdot,t)\|_\infty$$

Let $B(0,R) = \{u \in X_T : \|u\|_{X_T} \le R\}$ - closed ball in X_T, $R>0$ - fixed

we show that we can choose $T>0$, $R>0$ such that

$$\mathcal{F}: B(0,R) \to B(0,R)$$

and is a contraction

Step 1: $\mathcal{F}: B_R \longrightarrow B_R$

Let $u \in B_R$. we have to estimate $\mathcal{F}(u)$

przez części

$$|\mathcal{F}(u)(x,t)| \le \|u_0\|_\infty \underbrace{\int_{\mathbb{R}} G(y,t)\,dy}_{\overset{\shortparallel}{1}} + R^q \int_0^t \int_{\mathbb{R}} |G_y(x-y,t-s)|\,dy\,ds =$$

$$= \|u_0\|_\infty + R^q \int_0^t C_1 (t-s)^{-1/2}\,ds \le \|u_0\|_\infty + R^q C_2 T^{1/2} \le R$$

Step 2:

Contraction of $\mathcal{F}$ on B_R ⇧

Conclusion: For sufficiently large $R>0$ and sufficiently small $T>0$ we have a fixed point of $\mathcal{F}$ in B_R

Regularity: $\mathcal{F}: X_T \longrightarrow X_T$ has the property

$$\mathcal{F}: C^{2,1}(\mathbb{R} \times (0,T)) \to C^{2,1}(\mathbb{R} \times (0,T))$$

Conclusion: we have the local in time existance of solutions to (CD) for every $u_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$

Remark: The solution exists for $t \in [0,T]$ where $T = T(\|u_0\|_\infty)$

Remark: if $u_0 \in L^1(\mathbb{R})$ then $\mathcal{F}: C([0,T], L^1(\mathbb{R})) \to C([0,T], L^1(\mathbb{R}))$

Theorem: Assume that $u = u(x,t)$ is a solution of this problem for $t \in [0, T]$.
If $A \leq u_0(x) \leq B$, then $A \leq u(x,t) \leq B$ for all $x \in \mathbb{R}$ and $t \in [0, T]$.
(comparision theorem)

Lemma: Assume that $v = v(x,t)$ is a sufficiently good solution of

$$\begin{aligned} &v_t - v_{xx} + bv_x = 0 \\ &v(x,0) \leq 0 \end{aligned}$$

then $v(x,t) \leq 0$ for all $x \in \mathbb{R}$, $t \in [0,T]$

where $b = b(x,t) \subset C^{2,1}(\mathbb{R} \times (0,T))$, $b_x \in L^\infty(\mathbb{R} \times (0,T))$

Proof: STAMPACCHIA METHOD

we choose $\varphi \in C^2(\mathbb{R})$ with properties

$\varphi \geq 0$, $\varphi(s) = 0$ for $s \leq 0$

$\varphi(s) \sim s$ for $s > 1$ φ is convex, $\varphi'' \geq 0$

$$\frac{d}{dt} \int_{\mathbb{R}} \varphi(v(x,t))\, dx = \int_{\mathbb{R}} v_t \varphi'(v)\, dx = \int_{\mathbb{R}} v_{xx} \varphi'(v)\, dx - \int_{\mathbb{R}} b \underbrace{v_x \varphi'(v)}_{\varphi(v)_x} dx =$$

$$= - \int_{\mathbb{R}} |v_x|^2 \varphi''(v)\, dx + \int_{\mathbb{R}} b_x \varphi(v)\, dx \leq \|b_x\|_\infty \int_{\mathbb{R}} \varphi(v)\, dx$$

by the Gronwell's lemma:

$$\int_{\mathbb{R}} \varphi(v(x,t))\, dx \leq \underbrace{\int_{\mathbb{R}} \varphi(v(x,0))\, dx}_{\overset{\shortparallel}{0}}\, e^{t \|b_x\|_\infty}$$

so $\varphi(v(x,t)) = 0$

$v(x,t) \leq 0$ ■

Proof of the comparision theorem:

Let $v(x,t) = u(x,t) - B$

$$u_t - u_{xx} + qu^{q-1} u_x = 0$$

so $v_t - v_{xx} + qu^{q-1} v_x = 0$ applying lemna with $b = qu^{q-1}$

~~we get the thus~~: $b_x \in L^\infty$?

Theorem (Global existence):

For every $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ the solution of problem (CD) exists for all $t > 0$

Proof: By the comparision theorem for $t \in [0,T]$

$$-\|u_0\|_\infty \leq u(x,t) \leq \|u_0\|_\infty$$

we have a solution on $[0, T_1]$, $T_1 = T(\|u_0\|_\infty)$

as initial condition we take $u(x, T_1)$

we obtain solution on $t \in [T_1, 2T_1]$

$$\|u(x, T_1)\|_\infty \leq \|u_0\|_\infty$$

step by step: $[2T_1, 3T_1]$

$[3T_1, 4T_1]$ ...

Summary:

Theorem: For every $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ the problem (CD) has a solution $u \in C^{2,1}(\mathbb{R}\times(0,\infty))$ such that $u \in C([0,\infty), L^\infty(\mathbb{R})) \cap C([0,\infty), L^1(\mathbb{R}))$.
This is the unique solution with these properties

PROPERTIES OF SOLUTIONS:

1° If $u_0 \geqslant 0$, then $u(x,t) \geqslant 0$ for all $x \in \mathbb{R}$, $t \geqslant 0$
from now on, we assume that $u_0 \geqslant 0$ for simplicity of the exposition

2° Conservation of the integral

$$\forall t \geqslant 0 \qquad \int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_0(x)dx$$

from the Duhamel principle

$$u(x,t) = \int_{\mathbb{R}} G(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}} \partial_y G(x-y,t-s)u^q(y,s)dy\,ds \qquad / \int_{\mathbb{R}} \cdot\, dx$$

$$\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} [\underbrace{\int_{\mathbb{R}} G(x-y,t)dx}_{1}]u_0(y)dy + \underbrace{\int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} \dots dx\,dy\,ds}_{0}$$ because $\partial_y G(y,t)$ is odd

3° Decay estimates

$$\forall p \in [1,\infty] \quad \exists C(p) > 0 \qquad \|u(\cdot,t)\|_p \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})}\|u_0\|_1$$

Remark: w care $p=\infty$ only

$$\|u(\cdot,t)\|_\infty \leq Ct^{-\frac{1}{2}}\|u_0\|_1$$

Let $u = u(x,t)$ be a solution of (CD)

Rescaled family

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad \lambda > 0$$

Lemma 1: For every $0 < t_1 < t_2 < \infty$
$\{u_\lambda\}_{\lambda>0}$ is bounded in $C(\mathbb{R}\times[t_1,t_2])$

proof:

$$\|u_\lambda(\cdot,t)\|_\infty = \lambda\|u(\cdot,\lambda^2 t)\|_\infty \leq \lambda C(\lambda^2 t)^{-1/2}\|u_0\|_1 = Ct^{-1/2}\|u_0\|_1$$

$$\sup_{t_1 \leq t \leq t_2} \|u_\lambda(\cdot,t)\|_\infty \leq Ct_1^{-1/2}\|u_0\|_1$$

Goal: show that $\{\partial_x u_\lambda\}_{\lambda>0}$ is bounded in $C(\mathbb{R}\times[t_1,t_2])$

One possible way: we show that

$$\|\partial_x u(\cdot,t)\|_\infty \leq C(u_0)t^{-1}$$

$$\Rightarrow \|\partial_x u_\lambda(\cdot,t)\|_\infty \leq C(u_0)t^{-1}$$

Diffrent approach:

we write the equation for u_λ

$$u(x,t) = \frac{1}{\lambda}u_\lambda(\underbrace{\tfrac{x}{\lambda}}_{y}, \underbrace{\tfrac{t}{\lambda^2}}_{s})$$ we substitute to the equation

$$\lambda^{-1-2}\partial_s u_\lambda - \lambda^{-1-2}\partial_{yy}u_\lambda + \lambda^{-q-1}(u_\lambda^q)_y = 0$$

$$\partial_s u_\lambda - \partial_{yy}u_\lambda + \lambda^{2-q}\partial_y(u_\lambda^q) = 0$$

we come back to original variables

$$\partial_t u_\lambda - \partial_{xx}u_\lambda + \lambda^{2-q}\partial_x(u_\lambda^q) = 0$$

Duhamel principle

$$u_\lambda(t) = G(t) * u_{0,\lambda} - \lambda^{2-q}\int_0^t G(t-s) * (\partial_x u_\lambda^q)(s)\,ds$$

$$\partial_x u_\lambda(t) = \partial_x G(t) * u_{0,\lambda} - \lambda^{2-q}\int_0^t \partial_x G(t-s) * (q u_\lambda^{q-1}\partial_x u_\lambda)\,ds$$

we assume that $q \geqslant 2$ and $\lambda \geqslant 1$

$$\|\partial_x u_\lambda(t)\|_\infty \le C(t)\|u_{0,\lambda}\|_1 + C\int_0^t (t-s)^{-\beta}s^{-\gamma}\|\partial_x u_\lambda(s)\|_\infty\,ds$$

by Gronwall-like lemma

$$\|\partial_x u_\lambda(t)\|_\infty \le C(t, u_0) \qquad \text{no } \lambda$$

<u>Conclusion for today</u>

Under the assumption $\lambda \geqslant 1$ and $q \geqslant 2$

$$\left.\begin{aligned}\|u_\lambda(t)\|_\infty &\le C_1(t,\infty)\\ \|\partial_x u_\lambda(t)\|_\infty &\le C_2(t,u_0)\\ \|\partial_{xx} u_\lambda(t)\|_\infty &\le C_3(t,u_0)\end{aligned}\right\}\quad \text{no } \lambda$$

Hence from equation for u_λ

$$\|\partial_t u_\lambda(t)\|_\infty \le C_4(t,u_0)$$

<u>8.04. 2011</u>

<u>Theorem (comparison principle)</u>

Assume that $u \in C_b^{2,1}(\mathbb{R}\times[0,T])$ (everything is bounded)

and $\lim_{|x|\to\infty} u(x,t) = 0 \quad \forall t\in[0,T]$

Moreover assume that

$$\begin{cases} u_t - u_{xx} + b(x,t)u_x = 0 \\ u(x,0) \geqslant 0 \end{cases} \qquad (\text{no assumption on } b)$$

then $u(x,t) \geqslant 0$ for all $x\in\mathbb{R}$, $t\in[0,T]$

<u>Proof:</u> Define $f(t) = \cancel{\lim}\ \inf_{x\in\mathbb{R}} u(x,t)$ well-defined

Note that $f(0) = \inf_{x\in\mathbb{R}} u(x,0) = 0$ because $u(x,0)\geqslant 0$, $u(x,0)\to 0$, $|x|\to\infty$

Goal: show that $f(t) = 0$ for all $t\in[0,T]$

Suppose that there exists $t_0 < t_1$ s.t. $f(t_0) = 0$ and $f(t) < 0$ $t\in(t_0,t_1)$. For every $t\in(t_0,t_1)$ there exists $\xi(t)$ such that

$$f(t) = u(\xi(t), t)$$

<u>Fact 1:</u>

$f(t)$ is differentiable almost everywhere for $t\in(t_0,t_1)$

<u>Proof:</u> we show Lipschitz continuity. Let $s,t\in(t_0,t_1)$.

For $f(t) \leq f(s)$ we have

$$0 \leq f(s) - f(t) = \inf_x u(x,s) - u(\xi(t),t) \leq u(\xi(t), s) - u(\xi(t),t) \overset{\text{mean v. thm}}{\leq}$$

$$\leq |s-t| \cdot \sup_{\substack{x\in\mathbb{R}\\ t\in(t_0,t_1)}} |u_t(x,t)|$$

Now we apply Rademacher theorem (see Evans)

Fact 2: At each point of differentiability $f'(t) = u_t(\xi(t),t)$

(Fake proof: $f'(t) = \frac{d}{dt} u(\xi(t),t) = \xi'(t)\, \underset{\substack{\shortparallel\\ 0}}{u_x(\xi(t),t)} + u_t(\xi(t),t)$ because this is minimum

Correct proof: for $t, t+h \in (t_0,t_1)$, $h>0$

we have

$$f(t+h) = u(\xi(t+h), t+h) \leq u(\xi(t), t+h)$$

Hence

$$\frac{f(t+h) - f(t)}{h} \leq \frac{h(\xi(t), t+h) - u(\xi(t),t)}{h}$$

from below

$$\frac{f(t) - f(t-h)}{h} \geq \frac{u(\xi(t),t) - u(\xi(t), t-h)}{h}$$

For $h \to 0$, we complete the proof

Fact 3: There is $t_2 \in (t_0, t_1)$ where $f'(t_2) < 0$

Proof: $0 > f(t) = \int_{t_0}^{t} f'(s)ds$ (remember that $f(t_0) = 0$)

for $t \in (t_0, t_1)$ so we have t_2.

Conclusion:

$$0 > f'(t_2) = u_t(\xi(t_2), t_2) \overset{\text{from equation}}{=} \underset{\substack{\rotatebox{90}{$\leq$}\\ 0}}{u_{xx}}(\xi(t_2), t_2) - b(\xi(t_2), t_2)\underbrace{u_x(\xi(t_2), t_2)}_{0}$$

contradiction.

Back to the convection - diffusion equation

$$\begin{cases} u_t - u_{xx} + (u^q)_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Assumptions: $q>1$

$u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ $(\cap C^1(\mathbb{R}))$

Recall that under these assumptions we have a global-in-time "good" solution.

If $u_0 \geq 0$ then $u(x,t) \geq 0$ for all $x \in \mathbb{R}$, $t \geq 0$

Scaling

$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, $\lambda \geq 1$

this rescaled solution satisfies

$$\begin{cases} (u_\lambda)_t - (u_\lambda)_{xx} + \lambda^{2-q}(u_\lambda^q)_x = 0 \\ u_\lambda(x,0) = \lambda u_0(\lambda x) = u_{0,\lambda}(x) \end{cases}$$

For $q \geq 2$

$\|u_\lambda(\cdot,t)\|_\infty + \|\partial_t u_\lambda(\cdot,t)\|_\infty + \|\partial_x u_\lambda(\cdot,t)\|_\infty \leq C(t,u_0)$ - no λ

for $t>0$

Theorem:

There exists a sequence $\lambda_n \to \infty$ and a function $\bar{u} \in C(\mathbb{R}\times(0,\infty))$ ~~This is a uniform co~~ such that

$u_{\lambda_n}(x,t) \longrightarrow \bar{u}(x,t)$

This is a uniform convergence on every box $[-R,R]\times[t_1,t_2]$ for every $R>0$ and $0<t_1<t_2<\infty$

Proof: Apply the Arzela - Ascoli theorem and the diagonal argument by Cantor

Goal: There exists $t>0$

$\|u_{\lambda_n}(\cdot,t) - \bar{u}(\cdot,t)\|_\infty \xrightarrow{\lambda_n\to\infty} 0$

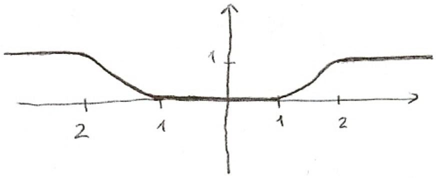
Tail's control

Let $\psi \in C^\infty(\mathbb{R})$

~~$\psi\uparrow 0$~~ ~~$\psi\nearrow 0$~~ $\psi(x) \geqslant 0$

$\psi(x) = 0$ for $|x| \leq 1$

$\psi(x) = 1$ for $|x| \geqslant 2$



define $\psi_R(x) = \psi\left(\frac{x}{R}\right)$

Lemma: For every $0<t_1<t_2<\infty$

$\sup\limits_{t_1\leq t\leq t_2} \|u_\lambda(\cdot,t)\psi_R(\cdot)\|_\infty \xrightarrow{R\to\infty} 0$ uniformly in λ

Proof of lemma:

Let $w(x,t) = u_\lambda(x,t)\,\psi_R(x)$

we write equation for w

$$w_t = (u_\lambda)_t\psi_R = (u_\lambda)_{xx}\psi_R + \lambda^{2-q}(u_\lambda^q)_x\psi_R = w_{xx} \underbrace{-2(u_\lambda)_x(\psi_R)_x}\,\underbrace{-u_\lambda(\psi_R)_{xx}}$$

$$+\lambda^{2-q}(u_\lambda^q\psi_R)_x \underbrace{-\lambda^{2-q}u_\lambda^q(\psi_R)_x}$$

$|(\psi_R)_x(x)| \leq \frac{C}{R}$

$|(\psi_R)_{xx}(x)| \leq \frac{C}{R^2}$

$|f(x,t,\lambda)| \leq C(t,u_0)$ ~~no~~$\to \left(1/_R + \frac{1}{R^2}\right)$

no λ!

so

$$\begin{cases} w_t = w_{xx} + f(x,t) + \lambda^{2-q}(u_\lambda^q\psi_R)_x \\ w(x,0) = w_0(x) \end{cases}$$

$u_\lambda^{q-1}\,\overset{w}{\widetilde{u_\lambda\cdot\psi_R}}$

$\shortparallel$

By the Duhamel principle

$$w(x,t) = G(t)*w_0 + \int_0^t G(t-s)*f(s)\,ds + \lambda^{2-q}\int_0^t G_x(t-s)*(u_\lambda^q\psi_R)\,ds$$

$$\|w(\cdot,t)\|_\infty \leq Ct^{-1/2}\|w_0\|_1 + C(t,u_0)\left(\frac{1}{R}+\frac{1}{R^2}\right) + C\int_0^t (t-s)^{-1/2}\|w(\cdot,s)\|_\infty\,ds$$

By the Gronwall lemma

$\|w(\cdot,t)\|_\infty \leq \left(Ct^{-1/2}\|w_0\|_1 + C(t)(R^{-1}+R^{-2})\right)C(t)$ no λ

Note that

$$\|w_0\|_1 = \|u_{0,\lambda}(\cdot)\psi_R(\cdot)\|_1 = \int_{\mathbb{R}} \lambda|u_0(\lambda x)\psi\left(\tfrac{x}{R}\right)|\,dx = \int_{\mathbb{R}} |u_0(x)\psi\left(\tfrac{x}{\lambda R}\right)|\,dx$$

$\downarrow R\to\infty$ uniformly in $\lambda\geqslant 1$

0

Theorem: There exists sequence $\lambda_n \to \infty$ s.t. for every $0 < t_1 < t_2 < \infty$

$$\sup_{t_1 \le t \le t_2} \| u_{\lambda_n}(\cdot, t) - \bar{u}(\cdot, t) \|_\infty \xrightarrow{\lambda_n \to \infty} 0$$

Proof: let $\varepsilon > 0$

$$\underbrace{\sup_{t_1 \le t \le t_2} \| (u_{\lambda_n}(\cdot, t) - \bar{u}(\cdot, t)(1 - \psi_R)(\cdot) \|_\infty}_{\xrightarrow{\lambda_n \to \infty} 0} + \underbrace{\sup_{t_1 \le t \le t_2} \| u_{\lambda_n}(\cdot, t) - \bar{u}(\cdot, t) \psi_R(\cdot) \|_\infty}_{\le \varepsilon}$$

for big R from lemma

because $\operatorname{supp}(1 - \psi_R) \subseteq [-2R, 2R]$

step 4: Identification of the limit

Lemma about initial conditions

For every $\varphi \in C_0^\infty(\mathbb{R})$

$$\left| \int_{\mathbb{R}} u_\lambda(x,t) \varphi(x) dx - \int_{\mathbb{R}} u_{0,\lambda} \varphi(x) dx \right| \le C(t) \xrightarrow{t \to \infty} 0 \quad \text{uniformly ea } \lambda \geqslant 1$$

Proof of lemma:

$$\frac{d}{dt} \int_{\mathbb{R}} u_\lambda \varphi \, dx = \int_{\mathbb{R}} (u_\lambda)_t \varphi \, dx = \int_{\mathbb{R}} (u_\lambda)_{xx} \varphi - \lambda^{2-q} \int_{\mathbb{R}} (u_\lambda^q)_x \varphi \, dx =$$

$$= \int u_\lambda \varphi_{xx} + \lambda^{2-q} \int_{\mathbb{R}} u_\lambda^q \varphi_x \, dx \le \| \varphi_{xx} \|_\infty \int u_\lambda + \| \varphi_x \|_\infty \| u_\lambda \|_q^q \le$$

$$\le C(t, \varphi, u_0)$$

$$\frac{d}{dt} \int u_\lambda \varphi \, dx \le C(t, \varphi, u_0) \quad \text{integrate}$$

$$\left| \int_{\mathbb{R}} u_\lambda(x,t) \varphi(x) - \int_{\mathbb{R}} u_\lambda(x,0) \varphi(x) dx \right| \le \int_0^t \underbrace{C(s, \varphi, u_0)}_{\text{integrable}} ds$$

Remark:

$$\int_{\mathbb{R}} u_{0,\lambda}(x) \varphi(x) dx = \int_{\mathbb{R}} \lambda u_0(\lambda x) \varphi(x) dx = \int_{\mathbb{R}} u_0(x) \varphi\left(\frac{x}{\lambda}\right) dx \xrightarrow[\text{by Lebesgue thm}]{\lambda \to \infty} \int u_0(x) dx \, \varphi(0)$$

Remark 2:

By Lebesgue theorem

$$\left| \int \bar{u}(x,t) \varphi(x) dx - \int u_0 \cdot \varphi(0) \right| \le C(t) \xrightarrow{t \to 0} 0$$

Theorem: Let $q > 2$

The function $\bar{u} \in C(\mathbb{R} \times (0, \infty))$ is a weak solution of

$$\begin{cases} \bar{u}_t = \Delta \bar{u} \\ \bar{u}(x,0) = \int_{\mathbb{R}} u_0 dx \, \delta_0 \end{cases}$$

Proof: $(u_\lambda)_t - (u_\lambda)_{xx} + \lambda^{2-q} (u_\lambda^q)_x = 0 \quad / \cdot \varphi \in C_0^\infty(\mathbb{R} \times [0, \infty)) \quad \int_0^\infty \int_{\mathbb{R}} dx dt$

$$- \int_0^\infty \int_{\mathbb{R}} u_\lambda \varphi_t \, dx dt - \int_{\mathbb{R}} u_{0,\lambda}(x) \varphi(x,0) dx - \int_0^\infty \int_{\mathbb{R}} u_\lambda \varphi_{xx} \, dx dt \overset{\to 0 \ (\lambda \to \infty)}{- \lambda^{2-q}} \int_0^\infty \int_{\mathbb{R}} u_\lambda^q \varphi_x \, dx dt = 0$$

$$-\int_0^\infty\int_{\mathbb{R}} \bar{u}\varphi_t \, dxdt - \int_{\mathbb{R}} u_0 dx\, \varphi(0) - \int_0^\infty\int_{\mathbb{R}} \bar{u}\varphi_{xx} \, dx dt = 0$$

there was a theorem - this initial value problem has a unique solution given by the formula $\bar{u}(x,t) = \int u_0 dx \, G(x,t)$

Corollary:

By the uniqueness of $G(x,t)$ the whole family

$\forall t>0$ $\quad u_\lambda(\cdot,t) \xrightarrow[\lambda\to\infty]{} \underbrace{\int u_0 dx}_{\overset{\shortparallel}{M}} \, G(\cdot,t)$ in $L^\infty(\mathbb{R})$

Corollary:

$$0 \xleftarrow{\lambda\to\infty} \| u_\lambda(\cdot,t_0) - MG(\cdot,t_0)\|_\infty = \lambda \| u(\cdot,\lambda^2 t_0) - M\, G(\cdot,\lambda^2 t_0)\|_\infty = C(t_0) t^{1/2} \| u(\cdot,t) - MG(\cdot,t)\|_\infty$$

Theorem:

Assume that u is a solution of the viscous Burgers equation

$$\begin{cases} u_t - u_{xx} + (u^2)_x = 0 \\ u(x,0) = u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \end{cases}$$

Then

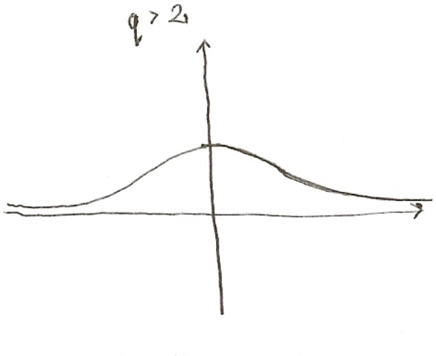
$\lambda u(\lambda x, \lambda^2 t) \xrightarrow{\lambda\to\infty} \bar{u}(x,t)$ in a reasonable sense

where $\bar{u}$ is a unique solution of

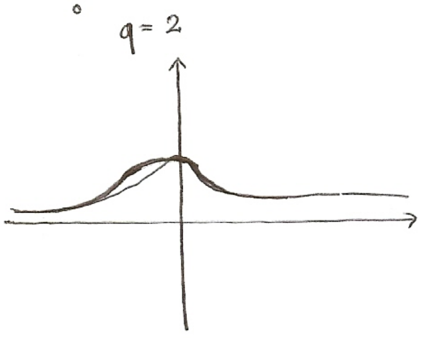
$$\begin{cases} \bar{u}_t - \bar{u}_{xx} + (\bar{u}^2)_x = 0 \\ \bar{u}(x,0) = M\delta_0 \end{cases}$$

Remark: This problem has an explicit solution

$$\bar{u}(x,t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{t}\right), \quad U(y) = \frac{e^{-y^2/4}}{K + \int_0^y e^{-z^2/4} dz}, \quad \int U = M$$



asymptotically weak non-linearity



equilibrium

$1 < q < 2$

Remark: The scaling $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ is not good

$$(u_\lambda)_t - (u_\lambda)_{xx} + \lambda^{2-q} (u_\lambda{}^q)_x = 0$$

Another scaling:

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^q t)$$

Lemma: This rescaled family satisfgies

$(u_\lambda)_t = \lambda^{q-2}(u_\lambda)_{xx} + (u_\lambda^q)_x = 0$

Now we apply the Four Steps Method

Step 1: Scaling $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^q t)$

Step 2: Estimates of u_λ independent of λ

Step 3: Passing to the limit

Step 4: Identification of the limit

Theorem: Let u be a nonnegative solution of

$u_t - u_{xx} + (u^q)_x = 0$ with $q \in (1,2)$

$u(x,0) = u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$

then $\lambda u(\lambda x, \lambda^q t) \xrightarrow{\lambda\to\infty} \bar{u}(x,t)$

in any reasonable sense (no in $L^\infty(\mathbb{R})$)

where $\bar{u}$ is a $\underset{\text{weak}}{\text{unique}}$ solution of

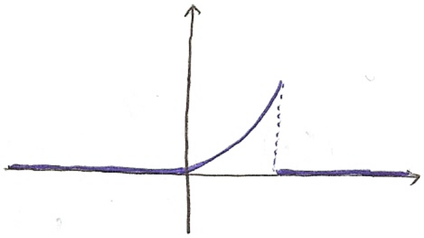
$$\begin{cases} u_t + (u^q)_x = 0 \\ u(x,0) = M\delta_0 \end{cases}$$

Here, nonlinearity is strong asymptotically.

Remark: The solution of this problem is explicit

$$\bar{u}(x,t) = \frac{1}{t^{1/q}} U\left(\frac{x}{t^{1/q}}\right)$$

$$U(y) = \begin{cases} c_1 y^{1/q-1}, & 0 < y < c_2 \\ 0, & \text{w.p.p.} \end{cases}$$



14.04.2011

Nieliniowe równanie ciepła

$$(\text{NC})\begin{cases} u_t = \Delta u + f(u) & x \in \mathbb{R}^n, \; t>0 \\ u(x,0) = u_0(x) \end{cases}$$

$f \in C^1(\mathbb{R})$ - zadana, $f(u) = \pm u^q$, $q > 1$

Tw (o istnieniu lokalnym rozwiązania)

Dla każdego $u_0 \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ istnieje $T = T(\|u_0\|_\infty)$ t.ze zagadnienie NC) ma rozwiązanie $u \in C^{2,1}(\mathbb{R}^n \times (0,T])$

w 2 (o przedłuzaniu rozwiązań)

zy założeniach tw 1 jezeli a priori

$\forall T>0 \sup_{0\le t\le T} \|u(t)\|_\infty < \infty$

o rozwiązanie u istnieje dla $t \in [0,\infty)$

w 3 (porównywanie rozwiązań)

łożmy, ze mamy 2 rozwiązania tw 1 $u, \tilde{u}$ t.ze $\forall x \in \mathbb{R}^n \; u_0(x) \le \tilde{u}_0(x)$.

ówczas $u(x,t) \le \tilde{u}_0(x,t)$ $\forall x \in \mathbb{R}^n$ $t \in (0,T)$.

Wniosek:

Jeżeli $f(0)=0$, to wtedy $u_0(x) \geq 0$ implikuje $u(x,t) \geq 0 \quad \forall x \in \mathbb{R}^n, t>0$

Dowód:

z tw. 3. $\tilde{u} \equiv 0$ jest rozwiązaniem

NIELINIOWE RÓWNANIE CIEPŁA Z ABSORBCJĄ

$$\begin{cases} u_t = \Delta u - u^q \\ u(x,0) = u_0(x) \geq 0 \end{cases} \qquad (NCA) \qquad \begin{matrix} x \in \mathbb{R}^n, t>0 \\ q>1 \end{matrix}$$

Pierwsze własności rozwiązań

1° $u(x,t) \geq 0$

2° wzór Duhamela

$$u(x,t) = G(t) * u_0(x) - \int_0^t \int_{\mathbb{R}^n} G(x-y, t-s) u^q(y,s)\, dy\, ds \leq G(t) * u_0(x)$$

Zatem

$u(x,t) \leq G(t) * u_0(x)$

$$\|u(\cdot,t)\|_\infty \leq \sup_x \left| \int G(x-y,t) u_0(y) dy \right| \leq \|u_0\|_\infty \sup_x \int_{\mathbb{R}^n} G(x-y,t) dy = \|u_0\|_\infty$$

Wniosek: Rozwiązanie istnieje dla wszystkich $t>0$

Krok 1. Skalowanie

$u_\lambda(x,t) = \lambda^n u(\lambda x, \lambda^2 t)$ – tak jak dla równania ciepła

Uwaga: Nie jest to jedyne skalowanie, które prowadzi do „czegoś". Jeszcze zobaczymy inne skalowanie.

Piszemy równanie na u_λ

$$u(x,t) = \frac{1}{\lambda^n} u_\lambda\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)$$

Podstawiamy do równania

$$\lambda^{-n-2} (u_\lambda)_2 = \lambda^{-n-2} \Delta_y u_\lambda - \lambda^{-nq} u_\lambda{}^q$$

$$(u_\lambda)_2 = \Delta u_\lambda - \lambda^{n+2-nq} u_\lambda{}^q$$

Będziemy badać asymptotykę $\lambda \to \infty$

ważne założenie $n+2-nq<0$

czyli $q > 1 + \frac{2}{n}$

Podstawowe założenia

$q > 1 + \frac{2}{2}$ WYKŁADNIK FUJITY

Asymptotycznie słaba nieliniowość.

Twierdzenie:

Zakładamy, że

- $q > 1 + \frac{2}{n}$
- $u_0 \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$
- $u_0 \geq 0$

Wówczas

$\forall t>0 \quad u_\lambda(\cdot,t) \xrightarrow{\lambda\to\infty} M_\infty \cdot G(\cdot,t) \quad$ w $\quad L^p(\mathbb{R}^n) \quad \forall p\in[1,\infty]$,

gdzie $\quad M_\infty = \int_{\mathbb{R}^n} u_0(y)\,dy - \int_0^\infty \int_{\mathbb{R}^n} u^q(y,s)\,dy\,ds$

<u>Uwaga:</u> $\quad 0 \le u(x,t) \le G(t)*u_0(x)$

$$\|u(\cdot,t)\|_q \le \|G(t)*u_0\|_q \overset{\text{nier. Yaunga}}{\le} \|G(t)\|_q \|u_0\|_1 = t^{-\frac{n}{2}(1-\frac{1}{q})} \|G(1)\|_q \|u_0\|_1$$

inne oszacowanie

$$\|u(\cdot,t)\|_q \le \|G(t)*u_0\|_q \le \|G(t)\|_1 \|u_0\|_q = \|u_0\|_q$$

Zatem

$$\int_{\mathbb{R}} u^q(y,s)\,dy = \|u(\cdot,s)\|_q^q \le \min\left\{ s^{-\frac{n}{2}(1-\frac{1}{q})\cdot q} \|G(1)\|_q^q \|u_0\|_1^q ,\ \|u_0\|_q^q \right\}$$

Zatem

$\int_0^\infty \int_{\mathbb{R}^n} u^q(y,s)\,dy\,ds \quad$ jest zbieżna , gdy $\quad -\frac{n}{2}(1-\frac{1}{q})q < -1$

$\Updownarrow \quad q > 1 + \frac{2}{n} \quad$ wykł. Fujity

<u>Krok 1:</u> Skalowanie

$u_\lambda(x,t) = \lambda^n u(x,t)$

<u>Krok 2:</u> oszacowania i zwartość

$0 \le u(x,t) \le G(t)*u_0(x)$

$\|u(\cdot,t)\|_p \le \|G(t)\|_p \|u_0\|_1 = Ct^{-\frac{n}{2}(1-\frac{1}{p})} \|u_0\|_1$

<u>Wniosek:</u>

- $\|u_\lambda(\cdot,t)\|_p \le Ct^{-\frac{n}{2}(1-\frac{1}{p})} \|u_0\|_1$ <u>na λ</u> $\quad \forall p\in[1,\infty)$
- Oszacowania na ∇u_λ

  $\|\nabla u_\lambda(\cdot,t)\|_p \le C(t,u_0)$, $t>0$ <u>na λ</u>

  stosujemy wzór Duhamela

<u>Twierdzenie (Arzela- Ascoliego)</u>

Niech $X\subset B\subset Y$ będą przestrzeniami Banacha t.że $X\subset B$ jest zwarte
$B\subset Y$ jest ciągłe

Załóżmy, że dla pewnego $p\in[1,\infty)$ i $T>0$ rodzina funkcji $\mathcal{F}$

1° $\mathcal{F}\subset L^p([0,T],X)$ jest ograniczona

2° $\{\partial_t f : f\in\mathcal{F}\} \subseteq L^p([0,T],Y)$ jest ograniczone

wówczas $\mathcal{F}\subset L^p([0,T],B)$ jest warunkowo zwarta

($\mathcal{F}\subset C([0,T],B)$ jest warunkowo zwarta jeżeli $p=\infty$)

<u>Uwaga:</u> Dla $p=\infty$, $X=Y=B=\mathbb{R}$ to jest tw. A-A.

stosujemy to twierdzenie z $B=L^p(K(0,R))$, $X=W^{1,p}(K(0,R))$,

$X\subset B$ zwarte , $\mathcal{F}=\{u_\lambda\}_{\lambda\ge 1}$ ograniczone w $L^\infty(t_1,t_2\}, W^{1,p}(K(0,R)))$

$\forall R>0 \quad \forall\, 0<t_1<t_2<\infty$

Jako Y bierzemy $Y = \cancel{W^{-1,1}(K(0,R))} = (W_0^{1,\infty}(K(0,R)))^*$

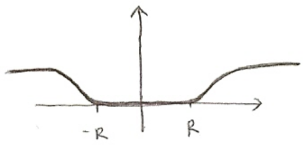
Niech φ będzie funkcją gładką o nośniku zwartym , $\varphi\in C_0^\infty(\mathbb{R}^n)$

$$\left|\int_{\mathbb{R}^n} \partial_t u_\lambda(x,t)\varphi(x)dx\right| = \left|\int_{\mathbb{R}^n}(\Delta u_\lambda - \lambda^{-nq+n+2}u_\lambda^{\,q})\varphi\right| \le \|\nabla\varphi\|_\infty \int_{\mathbb{R}^n}|\nabla u_\lambda| + \cancel{\|\varphi\|_\infty}$$

$$+\|\varphi\|_\infty \int_{\mathbb{R}^n} u_\lambda^{\,q}dx \le C(\|\varphi\|_\infty + \|D\varphi\|_\infty)$$

Oszacowanie ogonów:

$\sup_{t_1\le t\le t_2} \|u_\lambda(\cdot,t)\psi_R\|_p \xrightarrow{R\to\infty} 0$ jednostajnie względem λ.



Wniosek:

$\{u_\lambda\}_{\lambda\ge1}$ jest warunkowo zwarta w $C((0,\infty), L^p(\mathbb{R}^n))$, $\forall p\in[1,\infty]$.

Zatem istnieje $\bar{u}\in C((0,\infty), L^p(\mathbb{R}^n))$

Krok 3 i 4. Przejście graniczne i identyfikacja granicy

$(u_\lambda)_t = \Delta u_t - \lambda^{-nq+n+2}u_\lambda^{\,q} \quad / \cdot \varphi\in C_0^\infty(\mathbb{R}^n\times[0,\infty))$

$$-\int_0^\infty\int_{\mathbb{R}^n} u_\lambda\varphi_t\,dxdt - \int_{\mathbb{R}^n} u_\lambda(x,0)\varphi(x,0)dx = \int_0^\infty\int_{\mathbb{R}^n} u_\lambda\Delta\varphi\,dxdt \underbrace{- \lambda^{-nq+n+2}\int_0^\infty\int_{\mathbb{R}^n} u_\lambda^{\,q}\varphi\,dxdy}_{\downarrow\,\lambda\to\infty \atop 0}$$

w granicy widzimy słabe równanie ciepła

wn: $u_{\lambda^n} \xrightarrow{\lambda^n\to\infty} \bar{u}$ gdzie $\bar{u}$ jest słabym rozwiązaniem równania ciepła.

WARUNEK POCZĄTKOWY

$(u_\lambda)_t = \Delta u_\lambda - \lambda^{-nq+n+2}u_\lambda^{\,q} \quad / \varphi\in C^\infty(\mathbb{R})$

i całkujemy

$$\int_{\mathbb{R}^n} u_\lambda(x,t)\varphi(x)dx - \int_{\mathbb{R}^n} u_\lambda(x,0)\varphi(x)dx = \int_0^t\int_{\mathbb{R}^n} u_\lambda\Delta\varphi - \lambda^{-nq+n+2}\int_0^t\int_{\mathbb{R}^n} u_\lambda^{\,q}\varphi\,dxdy$$

$$\lambda^{-nq+n+2}\int_0^t\int_{\mathbb{R}^n}[\lambda^n u(\lambda x,\lambda^2 t)]^q\varphi(x)dxdt = \int_0^{\lambda^2 t}\int_{\mathbb{R}^n}(u(x,t))^q\varphi\left(\frac{x}{\lambda}\right)dxdt$$

Zatem

$$\left|\int_{\mathbb{R}^n} u_\lambda(x,t)\varphi(x)dx - \int_{\mathbb{R}^n} u_\lambda(x,0)\varphi(x)dx + \int_0^\infty\int_{\mathbb{R}^n} u^q(x,t)\varphi\left(\frac{x}{\lambda}\right)dxdt\right| \le \Theta(1) \quad \begin{matrix} t\to0 \\ \text{jedn. wzgl. } \lambda\end{matrix}$$

Zostało udowodnione

$\lambda^n u(\lambda\cdot,\lambda^2 t) \xrightarrow{\lambda\to\infty} M_\infty G(\cdot,t)$ zbieżność w $L^p(\mathbb{R}^n)$

równoważne sformułowanie

$t^{\frac{n}{2}(1-\frac{1}{p})}\|u(\cdot,t) - M_\infty G(\cdot,t)\|_p \xrightarrow{t\to\infty} 0$

przy założeniu, że $q > 1+\frac{2}{n}$. Co będzie gdy $q < 1+\frac{2}{n}$?

Uwaga: ◦ $q \in (1, 1+\frac{2}{n})$

Równanie $u_t = \Delta u - u^q$

rozwiązaniami tego równania są funkcje $f = f(t)$ spełniające

$$\begin{cases} f' = -f^q \\ f(0) = f_0 \end{cases}$$

Rozwiązaniem jest $f(t) = \left(f_0^{1-q} + t(q-1)\right)^{-\frac{1}{q-1}}$ ta funkcja maleje jak $t^{-\frac{1}{q-1}}$

Pokazaliśmy, że dla $q > 1+\frac{2}{n}$

$\|u(\cdot,t)\|_\infty \sim C t^{-n/2}$

$-\frac{1}{q-1} < -\frac{n}{2}$ dla $q < 1+\frac{2}{n}$

Wniosek: Dla $q < 1+\frac{2}{n}$ rozwiązanie maleje szybciej niż $G(\cdot, t)$

Nieliniowe równanie ciepła z reakcją

$$\begin{cases} u_t = \Delta u + u^q \\ u(x,0) = u_0(x) \geq 0 \end{cases} \qquad q > 1$$

Wzór Duhamela

$$u(x,t) = G(t) * u_0(x) + \int_0^t \int_{\mathbb{R}^n} G(x-y, t-s)\, u^q(y,s)\, dy\, ds \geq G(t) * u_0(x)$$

czyli $u(x,t) \geq G(t) * u_0(x)$

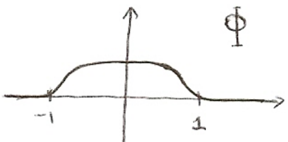
Metoda Pochozajewa

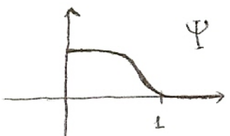
Załóżmy, że mamy gładkie w czasie, słabe rozwiązanie $u(x,t) > 0$ tego zagadnienia

$$-\int_0^\infty \int_{\mathbb{R}^n} u(x,t)\varphi_t(x,t)\,dx\,dt - \int_{\mathbb{R}^n} u_0(x)\varphi(x,0)\,dx = \int_0^\infty \int_{\mathbb{R}} u\Delta\varphi\,dx\,dt + \int_0^\infty \int_{\mathbb{R}^n} u^q\varphi\,dx\,dt$$

$$\int_0^\infty \int_{\mathbb{R}^n} u^q\varphi\,dx\,dt + \int_{\mathbb{R}^n} u_0(x)\varphi(x,0)\,dx = -\int_0^\infty \int_{\mathbb{R}^n} u(\varphi_t + \Delta\varphi)\,dx\,dt$$

$$\varphi(x,t) = \Phi\left(\frac{|x|}{R}\right)\Psi\left(\frac{t}{R^2}\right)$$





$$\left|-\int_0^\infty \int_{\mathbb{R}^n} \frac{u}{\varphi^{1/q}}\left[\varphi^{1/q}(\varphi_t + \Delta\varphi)\,dx\,dt\right]\right| \leq \qquad (a\cdot b \leq \varepsilon a^q + C(\varepsilon)b^{q'})$$

$$\leq \varepsilon \int_0^\infty \int_{\mathbb{R}^n} u^q\varphi\,dx\,dt + C(\varepsilon)\int_0^\infty \int_{\mathbb{R}^n} \left|\frac{\varphi_t + \Delta\varphi}{\varphi^{1/q}}\right|^{q'} dx\,dt$$

Stąd oszacowanie

$$(1-\varepsilon)\int_0^\infty \int_{\mathbb{R}^n} u^q\varphi\,dx\,dt + \int_{\mathbb{R}^n} u_0(x)\varphi(x,0)\,dx \leq C(\varepsilon)\int_0^\infty \int_{\mathbb{R}^n} \left|\frac{\varphi_t + \Delta\varphi}{\varphi^{1/q}}\right|^{q'} dx\,dt$$

Podstawiamy konkretne φ

$$\int_0^\infty \int_{\mathbb{R}^n} \left|\frac{\varphi_t + \Delta\varphi}{\varphi^{1/q}}\right|^{q'} dx\,dt = R^{-2q'+n+2} \cdot \underset{\uparrow}{C}$$

no R [illegible]

Lemat: $-2q' + n + 2 < 0$ dla $q < 1 + \frac{2}{n}$

Przechodząc z $R \to \infty$ dostajemy $u \equiv 0$ dla $q < 1 + \frac{2}{n}$

Zadanie na piątek

Niech $q > 1 + \frac{2}{n}$. Jeżeli $u_0 \in L^1 \cap L^\infty(\mathbb{R}^n)$ jest dostatecznie mały, to rozwiązanie jest globalne, duży, to rozwiązanie nie jest globalne.

15.04.2011

Self-similar solutions of the heat-equation

Recall Scaling $\lambda^n u(\lambda x, \lambda^2 t)$, $x \in \mathbb{R}^n$

Remark Gauß-Weierstraß kernel

$$G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$$

is invariant under this scaling

$$\lambda^n G(\lambda x, \lambda^2 t) = G(x,t)$$

Def: A solution is self-similar if it is invariant for some scaling.

The Cauchy problem

$$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n \\ u(x,0) = u_0(x) \end{cases}$$

Solution

$$u(x,t) = \int_{\mathbb{R}^n} G(x-y, t) u_0(y)\,dy$$

Let $u_0 = \frac{1}{|x|^a}$, $a \in (1, n)$

Lemma: The function

$\forall\, t > 0$ $\quad u(x,t) = \int_{\mathbb{R}^n} G(x-y, t) \frac{1}{|y|^a}\,dy \in L^p(\mathbb{R}^n)$ with $p \in (\frac{n}{a}, \infty]$

and $u \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$ and satisfies the heat equation $u_t = \Delta u$.

Proof: The Young inequality

$$\|f * g\|_p \le \|f\|_q \cdot \|g\|_r$$

$$\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$$

we decompose

$$\int_{\mathbb{R}^n} G(x-y,t) \frac{1}{|y|^a}\,dy = \int_{|y| \le 1} G(x-y,t) \underset{\substack{\cap \\ L^r(\mathbb{R}^n) \\ r < \frac{n}{a}}}{\frac{1}{|y|^a}}\,dy + \int_{|y| \ge 1} G(x-y,t) \underset{\substack{\cap \\ L^r(\mathbb{R}^n) \\ r > \frac{n}{a}}}{\frac{1}{|y|^a}}\,dy$$

we apply the Young inequality

Corollary:

$u(x,t) = \int_{\mathbb{R}^n} G(x-y, t) \frac{1}{|y|^a}\,dy$ is a self-similar solution of the heat equation.

Proof: we look for scaling. we change variables

$$u(x,t) = \int_{\mathbb{R}^n} t^{-n/2} G\left(\frac{x-y}{\sqrt{t}}, 1\right) \frac{1}{|y|^a}\, dy = \{ y = \sqrt{t} z \} = \int_{\mathbb{R}^n} G\left(\frac{x}{\sqrt{t}} - z, 1\right) \frac{1}{|\sqrt{t} z|^a}\, dz =$$

$$= t^{-a/2}\, u_a\left(\frac{x}{\sqrt{t}}\right)$$

where $u_a(y) = \int_{\mathbb{R}^n} G(y-z)\, \frac{1}{|z|^a}\, dz$, $u_a \in L^p(\mathbb{R}^n)$, $p \in (\frac{n}{a}, \infty]$

So, the scaling is

$$\lambda^a u(\lambda x, \lambda^2 t) = u_\lambda(x,t)$$

This solution is invariant under this scaling.

Remark: If u_0 is a homogenous function of degree $-a$ for some $a \in (0,n)$. Then the corresponding solution of the heat equation is self-similar.

~~Let~~ ~~$u_0(x) = \frac{1}{|x|^a}$~~ ~~$a \in (1,n)$~~

Remark: we assume that $a \in (0,n)$ to have $u_0 \in L^1_{loc}(\mathbb{R}^n)$

Theorem: (Self-similar large time behaviour)

Assume that $u_0 \in L^\infty(\mathbb{R}^n)$ satisfies

$$\lim_{|x|\to\infty} |x|^a u_0(x) = \underset{\text{constant}}{K} \neq 0 \qquad (H1)$$

for some $a \in (0,n)$ and a constant $K \in \mathbb{R}$

Let u be the corresponding solution of the heat equation.

Then

$$t^{a/2} \| u(\cdot,t) - K t^{-a/2} u_a\left(\frac{\cdot}{\sqrt{t}}\right) \|_\infty \xrightarrow{t\to\infty} 0$$

Remark: This is equivalent to $(\forall t_0 > 0)$

$$\lambda^a u(\lambda\cdot, \lambda^2 t) \xrightarrow{\lambda\to\infty} K t_0^{-a/2} u_a\left(\frac{x}{\sqrt{t_0}}\right) \quad \text{in} \quad L^\infty(\mathbb{R}^n)$$

Proof: we estimate

$$u(x,t) - K t^{-a/2} u_a\left(\frac{x}{\sqrt{t}}\right) = \int_{\mathbb{R}^n} t^{-n/2} G\left(\frac{x-y}{\sqrt{t}}, 1\right) \left[u_0(y) - \frac{K}{|y|^a}\right] dy = \{ y = \sqrt{t} w \}$$

$$= \int_{\mathbb{R}^n} G\left(\frac{x}{\sqrt{t}} - y, 1\right) \Big[\underbrace{t^{a/2} u_0(\sqrt{t} w) - \frac{K}{|w|^a}}_{\downarrow\, t\to\infty \atop 0} \Big] dy$$

by the assumption on u_0.

By Lebesgue* we complete the proof.

Remark:

Assumption (H1) implies $u_0 \notin L^1(\mathbb{R}^n)$

Nonlinear heat equation

$$\begin{cases} u_t = \Delta u \pm u^q, & x \in \mathbb{R}^n \\ u(x,0) = u_0(x) \end{cases}$$

we know:

1° $\forall u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

we have local in time solution

2° $u_0 \geq 0 \Rightarrow u(x,t) \geq 0$

3° Fujita exponent

$$q_F = 1 + \frac{2}{n}$$

4° Equation with absorbtion "−"

For $q > q_F$, a solution looks as the heat kernel when $t \to \infty$

For $1 < q < q_F$, there are solutions which decay faster then $G(x,t)$

5° Equation with reaction "+"

For $1 < q < q_F$, each solution blows up in finite time

Today: $q > q_F = 1 + \frac{2}{n}$

For simplicity

$$\begin{cases} u_t = \Delta u \pm u^3, & x \in \mathbb{R}^3 \\ u(x,0) = u_0(x) \geq 0 \end{cases}$$

$$q_F = 1 + \frac{2}{3}$$

Scaling

If $u(x,t)$ is a solution when $\lambda u(\lambda x, \lambda^2 t)$ is also a solution to this equation.

More generally

$$u_t = \Delta u \pm u^q$$

Remark: if $u(x,t)$ is a solution when $\lambda^{\frac{2}{q-1}}(\lambda x, \lambda^2 t)$ is also a solution to this equation.

Duhamel formula:

$$u(x,t) = G(t) * u_0 \pm \int_0^t \int_{\mathbb{R}^n} G(x-y, t-s) u^3(y,s)\, dy ds \qquad \text{(DUH)}$$

Abstract lemma:

Consider the equation $x = x_0 + B(x,x,x)$ on a Banach space X

3-linear form

Assume that

1° $x_0 \in X$ is given

2° $\exists\, C_0 > 0 \quad \forall x_1, x_2, x_3 \quad \|B(x_1,x_2,x_3)\|_X \leq C_0 \|x_1\|_X \|x_2\|_X \|x_3\|_X$

then there exists $C_1 = C_1(C_0)$ s.t.

if $\|x_0\| \leq C_1$ then the equation has a unique solution

in $B(0, \frac{C_1}{2})$

Example: $X = \mathbb{R}$, $x = x_0 + C x^3$

The Kato- weissler space

$$X = C_b([0,\infty), L^3(\mathbb{R}^3)) \cap \{u \in C((0,\infty), L^6(\mathbb{R}^3)) : \sup_{t>0} t^{1/4} \|u(t)\|_6 < \infty\}$$

with the norm

$$\|u\| = \sup_{t>0} \|u(t)\|_3 + \sup_{t>0} t^{1/4} \|u(t)\|_6$$

Remark:

if $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ then

$\|u_\lambda\| = \|u\|$

$$\sup_{t>0} \|u_\lambda(\cdot,t)\|_3 = \sup_{t>0} \lambda^{1-\frac{3}{3}} \|u(\cdot,\lambda^2 t)\|_3 = \sup_{t>0} \|u(\cdot,t)\|_3$$

Lemma 1: If $u_0 \in L^3(\mathbb{R}^3)$, then $G(t) * u_0 \in X$

Proof: $\|G(t) * u_0\|_3 \leq \|G(t)\|_1 \cdot \|u_0\|_3 = \|u_0\|_3$

$$p = \frac{6}{5}$$

$$\|G(t) * u_0\|_6 \leq \|G(t)\|_{\frac{6}{5}} \cdot \|u_0\|_3 = c\, t^{-\frac{3}{2}(1-\frac{1}{p})} \|u_0\|_3 = c\, t^{-1/4} \|u_0\|_3$$

Corollary: $\|G(t) * u_0\|_X \leq C \|u_0\|_3$

Lemma 2: Let $B(u_1, u_2, u_3) = \pm \int_0^t \int_{\mathbb{R}^n} G(x-y, t-s)(u_1 \cdot u_2 \cdot u_3)(y, s)\, dy\, ds$

There exists $C_1 > 0$ s.t. for all $u_1, u_2, u_3 \in X$ we have

$\|B(u_1, u_2, u_3)\| \leq C_1 \|u_1\| \cdot \|u_2\| \cdot \|u_3\|$

Proof: Hölder inequality

$$\|u_1(s) \cdot u_2(s) \cdot u_3(s)\|_2 \leq \|u_1(s)\|_6 \cdot \|u_2(s)\|_6 \cdot \|u_3(s)\|_6 \leq s^{-3/4} \|u_1\|_X \cdot \|u_2\|_X \cdot \|u_3\|_X$$

step 2:

$$\left\| \int_0^t G(t-s) * (u_1(s) \cdot u_2(s) \cdot u_3(s))\, ds \right\|_3 \overset{\text{Young}}{\leq} C \int_0^t (t-s)^{-1/4} \|u_1 \cdot u_2 \cdot u_3\|_2\, ds \leq$$

$$\leq C \underbrace{\int_0^t (t-s)^{-1/4} s^{-3/4} ds}_{\substack{\| \; \{s = t\tau\} \\ \text{const}}} \|u_1\| \cdot \|u_2\| \cdot \|u_3\|$$

step 3:

$$t^{1/4} \left\| \int_0^t G(t-s) * u_1 u_2 u_3 (s)\, ds \right\|_6 \leq C t^{1/4} \int_0^t (t-s)^{-1/2} \|u_1(s) u_2(s) u_3(s)\|_2\, ds \leq$$

$$\leq C t^{1/4} \underbrace{\int_0^t (t-s)^{-1/2} s^{-3/4} ds}_{\text{~~idepenbenb~~dent of } t} \|u_1\|_X \cdot \|u_2\|_X \cdot \|u_3\|_X$$

Applying the abstract lemma we have

Theorem: There exists $C_1 > 0$ s.t. for all $u_0 \in L^3(\mathbb{R}^3)$ s.t. $\|u_0\|_3 < C_1$ the equation (DNH) has a solution in X. This is the unique solution in $B(0, \frac{C_2}{2}) \subset X$.

Remark: One can show that this function $u = u(x,t) \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$ is a solution of

$$\begin{cases} u_t = \Delta u \pm u^3 & x \in \mathbb{R}^3 \\ u(x,0) = u_0(x) \end{cases}$$

General case:

$$\begin{cases} u_t = \Delta u \pm u^q & , x \in \mathbb{R}^n \\ u(x,0) = u_0(x) \end{cases}$$

scaling $\lambda^{\frac{2}{q-1}} u(\lambda x, \lambda^2 t)$

The space X: $\sup_{t>0} \|u(t)\|_p = \sup_{t>0} \lambda^{\frac{2}{q-1} - \frac{n}{p}} \|u(\cdot, \lambda^2 t)\|_p$

we need $\frac{2}{q-1} - \frac{n}{p} = 0 \Rightarrow \frac{2}{q-1} = \frac{n}{p} \Rightarrow p = \frac{n(q-1)}{2} \geq 1$

when $q \geq 1 + \frac{2}{n}$

Back to $\begin{cases} u_t = \Delta u \pm u^3 & , x \\ u(x,0) = u_0(x) \end{cases}$

Assume that u_0 is self-similar

$u(x,t) = \lambda u(\lambda x, \lambda^2 t) \quad \forall \lambda > 0 \quad \forall x \in \mathbb{R}^3 \quad \forall t > 0$

Let ~~$\lambda > 0$~~ $\lambda = \frac{1}{\sqrt{t}}$

Then $u(x,t) = \frac{1}{\sqrt{t}} u(\frac{1}{\sqrt{t}} x, \frac{t}{t} 1)$

Initial conditions

Assume that

$\lim_{t \to 0} u(x,t) = u_0(x)$ exists

Lemma: Then $u_0(x)$ has to be ~~non~~homogenous of degree -1.

Proof: we have to show that $\forall K > 0 \quad \forall x \quad K u_0(Kx) = u_0(x)$

$$K u_0(Kx) = \lim_{t \to 0} K u(Kx, t) \overset{\text{self-similar}}{=} \lim_{t \to 0} K \frac{1}{\sqrt{t}} u(\frac{Kx}{\sqrt{t}}, 1) = \lim_{t \to 0} \frac{1}{\sqrt{t}} u(\frac{x}{\sqrt{t}}, 1) = u_0(x)$$

Remark:

For self-similar solutions

$u_0(x) \sim \frac{1}{|x|} \notin L^3(\mathbb{R}^3)$

Back to Duhamel

$$u(x,t) = G(t) * u_0 \pm \int_0^t G(t-s) u^3(s) ds$$

we know

$\forall t > 0, \quad G(t) * \frac{1}{|x|} \in L^6(\mathbb{R}^3)$

Conclusion:

Assume that $u_0(x) = \frac{\varepsilon}{|x|}$.

If ε is small enough, then the Duhamel equation has a solution in $\tilde{X} = \{u \in C((0,\infty), L^6(\mathbb{R}^3)) : \|u\| = \sup_{t>0} t^{1/4} \|u(t)\|_6 < \infty\}$

this is unique solution in $B(0,\eta)$ for small $\eta > 0$

Remark: This solution is self-similar i.e.

$$\lambda u(\lambda x, \lambda^2 t) = u(x,t)$$

Proof:

- Let $w(x,t) = \lambda u(\lambda x, \lambda^2 t)$. Note that $w(x,0) = \lambda u(\lambda x, 0) = \frac{\lambda \varepsilon}{|\lambda x|} = \frac{\varepsilon}{|x|} = u_0(x)$
- w is a solution
- w is small (because all norms are invariant)

By uniqueness $u_\lambda(x,t) = u(x,t)$

Theorem (self-similar asymptotics)

Let $\varepsilon > 0$ be sufficiently small. Denote by $U(x,t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right)$ a self-similar solution with the initial condition $U_0(x) = \frac{\varepsilon}{|x|}$

Assume that $u_0 \in L^\infty(\mathbb{R}^3)$ satisfies

$$\lim_{|x| \to \infty} |x| u_0(x) = \varepsilon$$

Then $t^{1/4} \| u(\cdot,t) - \frac{1}{\sqrt{t}} U\left(\frac{\cdot}{\sqrt{t}}\right) \|_6 \xrightarrow{t \to \infty} 0$

Proof: Duhamel

$$u(t) = G(t) * u_0 \pm \int_0^t G(t-s) u^3(s) ds$$

$$U(t) = G(t) * U_0 \pm \int_0^t G(t-s) U^3(s) ds \qquad \nearrow \overset{t\to\infty}{0} \text{ done}$$

$$t^{1/4} \| u(t) - U(t) \|_6 \leq t^{1/4} \| G(t) * (u_0 - U_0) \|_6 + \underbrace{t^{1/4} \int_0^t \| G(t-s) * (u^3(s) - U^3(s)) \|_6 ds}_{(*)}$$

$$(*) \leq C t^{1/4} \int_0^t (t-s)^{-1/2} s^{-3/4} \left(s^{1/4} \| u(s) - U(s) \|_6 \right) ds \cdot \left(\|u\|_X^2 + \|u\|_X \|U\|_X + \|U\|_X \right)$$

so $f(t) := t^{1/4} \| u(t) - U(t) \|_6$

$$f(t) \leq g(t) + C \int_0^t (t-s)^{-1/2} s^{-3/4} f(s) ds \cdot C(\|u\|, \|U\|)$$

Lemma: If $g(t) \to 0$ $t \to \infty$ and if $C(\|u\|, \|U\|)$ is small then $f(t) \to 0$, $t \to \infty$

Proof: Excersize

19.05.2011

układ Naviera-Stokesa w $\mathbb{R}^3$

$$\begin{cases} u_t - \Delta u + (u \circ \nabla) u + \nabla p = F \\ \nabla \circ u = 0 \end{cases}$$

$u = u(x,t) = (u_1, u_2, u_3)$

$p = p(x,t)$

$x \in \mathbb{R}^3$, $t > 0$

F - dane

przykładając $\nabla \circ$ do pierwszej równości otrzymujemy

$$\nabla \circ (u \circ \nabla) u + \Delta p = \nabla \circ F$$

formalnie $p = (-\Delta)^{-1} \left(\nabla \circ (u \circ \nabla) u - \nabla \circ F \right)$

Cel wykładu:

Równanie reakcji dyfuzji (CD)

$$\begin{cases} u_t - \Delta u + (a\circ\nabla)u^2 = F \\ u(x,0) = u_0(x) \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^3 \\ t>0 \\ u = u(x,t) \in \mathbb{R} \end{array}$$

Formal argument. Compute Fourier transform

$$\hat{u}_t(\xi,t) + |\xi|^2\hat{u}(\xi,t) + i(a\circ\xi)\hat{u}^2(\xi,t) = \hat{F}(\xi,t) \quad | \, e^{t|\xi|^2}$$

$$\left(\hat{u}(\xi,t)e^{t|\xi|^2}\right)_t + i(a\circ\xi)e^{t|\xi|^2}\hat{u}^2(\xi,t) = e^{t|\xi|^2}\hat{F}(\xi,t) \quad \Big| \int_0^t$$

we obtain the formula (Int. Eq)

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)e^{-t|\xi|^2} - \int_0^t e^{-(t-s)|\xi|^2} i(a\circ\xi)\hat{u}^2(\xi,s)\,ds + \int_0^t e^{-(t-s)|\xi|^2}\hat{F}(\xi,s)\,ds$$

Def: By a solution of problem (CD) we understand the solution of this integral equation

Definition of the space

$$\mathcal{PM}^a = \{ f \in S'(\mathbb{R}^n) : \hat{f} \in L^1_{loc}(\mathbb{R}^n), \ \|f\|_a = \operatorname*{esssup}_{\xi} |\xi|^a |\hat{f}(\xi)| < \infty \}$$

here always $a \geq 0$

Examples:

1) $f \in S(\mathbb{R}^n)$, $C_0^\infty(\mathbb{R}^n)$ $\Rightarrow$ $f \in \mathcal{PM}^a$ $\forall a \geq 0$

2) Dirac delta

$\delta_0 \in S'(\mathbb{R}^3)$, $\hat{\delta}_0(\xi) = C$ - constant

$\delta_0(\varphi) = \varphi(0)$ this means $\delta_0 \in \mathcal{PM}^0 \setminus \mathcal{PM}^a$, $a>0$

3) $f(x) = \dfrac{1}{|x|^b}$ $\quad b \in (0,3)$

$\hat{f}(\xi) = \dfrac{C}{|\xi|^{3-b}}$

$f \in \mathcal{PM}^{3-b}$

Terminology:

$\mathcal{PM}^0 = \mathcal{PM}$ (pseudomeasures)

f is a pseudomeasure if $\hat{f} \in L^\infty(\mathbb{R}^3)$

Excercize: Every finite measure on $\mathbb{R}^3$ is a pseudomeasure

4) Let $f \in S'(\mathbb{R}^3)$ such that

$\hat{f}(\xi) = \dfrac{i\xi_1}{|\xi|} \in L^\infty(\mathbb{R}^3)$ Riesz transform

so $f \in \mathcal{PM}$ but f is not a measure, because $\hat{f}(\xi)$ is not continuous

Scaling property of PM^a
of the norm of PM^a

For $f \in S(\mathbb{R}^3)$ we define $f_\lambda(x) = f(\lambda x) \quad \forall \lambda > 0$

we want to compute $\| f_\lambda \|_a$

<u>Lemma:</u> For $f_\lambda(x) = f(\lambda x)$

$$\| f_\lambda \|_a = \lambda^{a-3} \| f \|_a$$

<u>Proof:</u>

$$\hat{f}_\lambda(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\xi x} f(\lambda x) dx = \lambda^{-3} (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\xi \frac{z}{\lambda}} f(z) dz =$$

$$= \lambda^{-3} \hat{f}\left(\frac{\xi}{\lambda}\right)$$

Now

$$\| f_\lambda \|_a = \operatorname*{ess\,sup}_{\xi \in \mathbb{R}^3} \lambda^a \left| \frac{\xi}{\lambda} \right|^a \lambda^{-3} \left| \hat{f}\left(\frac{\xi}{\lambda}\right) \right| = \lambda^{a-3} \operatorname*{ess\,sup}_{\xi \in \mathbb{R}^3} \left| \frac{\xi}{\lambda} \right|^a \left| \hat{f}\left(\frac{\xi}{\lambda}\right) \right| =$$

$$= \lambda^{a-3} \| f \|_a .$$

<u>Excersize:</u> For every $a \geq 0$ $(PM^a, \| \cdot \|_a)$ is a Banach space.

<u>Heat equation on PM^a spaces</u>

Notation $S(t)u_0 = G(\cdot, t) * u_0$ # $G(x,t) = (4\pi t)^{-3/2} e^{-|x|^2/4t}$

<u>Lemma:</u> Let $a \geq 0$

For every $u_0 \in PM^a$, we have $S(t)u_0 \in PM^a$, $\forall t > 0$

Moreover
$$\| S(t)u_0 \|_a \leq \| u_0 \|_a$$

<u>Proof:</u> we use well-known fact

$$\hat{G}(\xi, t) = e^{-t|\xi|^2}$$

$$\widehat{G(\cdot,t) * u_0}(\xi) = \hat{G}(\xi, t)\, \hat{u}_0(\xi)$$

then
$$\| S(t)u_0 \|_a = \operatorname*{ess\,sup}_{\xi} |\xi|^a |\widehat{S(t)u_0}(\xi)| = \operatorname*{ess\,sup}_{\xi} |\xi|^a |e^{-t|\xi|^2} \hat{u}_0(\xi)| \leq$$

$$\leq \operatorname*{ess\,sup}_{\xi} |\xi|^a |\hat{u}_0(\xi)| = \| u_0 \|_a$$

<u>Lemma:</u> Let $a \geq 0$

Assume that $F \in L^\infty([0,\infty), PM^a)$

then $\int_0^t S(t-s) F(s) ds \in L^\infty([0,\infty), PM^{a+2})$

<u>Proof:</u>

we estimate

$$|\xi|^{a+2} \left| \int_0^t e^{-(t-s)|\xi|^2} \hat{F}(\xi, s) ds \right| = \left| |\xi|^2 \int_0^t e^{-(t-s)|\xi|^2} |\xi|^a \hat{F}(\xi, s) ds \right| \leq$$

$$\leq \left| |\xi|^2 \int_0^t e^{-(t-s)|\xi|^2} ds \right| \| F \|_{L^\infty(0,\infty, PM^a)}$$

$$|\xi|^2 \int_0^t e^{-z|\xi|^2} dz = 1 - e^{-t|\xi|^2} \le 1$$

$$\left\| \int_0^t S(t-s)F(s)\,ds \right\|_{L^\infty([0,\infty),\, PM^{a+2})} \le \|F\|_{L^\infty((0,\infty),\, PM^a)}$$

Remark on the definition of solutions:

we think about the equation (Int. Eq) in the following way

$$u(t) = S(t)u_0 - \int_0^t S(t-s)(a\circ\nabla)u^2(s)\,ds + \int_0^t S(t-s)F(s)\,ds$$

Remark:

Heat equation

$$\begin{cases} u_t = \Delta u \\ u(x,0) = u_0(x) \end{cases}$$

Fourier transform

$$\hat u_t(\xi,t) = -|\xi|^2 \hat u(\xi,t)$$

solution $\hat u(\xi,t) = e^{-t|\xi|^2} \hat u_0(\xi,t)$

Estimates on the nonlinear term:

we define the bilinear form

$$B(u,v)(t) = \int_0^t S(t-s)(a\circ\nabla)[u(s)v(s)]\,ds$$

Lemma:

$\exists\, \eta > 0 : \forall\, u,v \in L^\infty([0,\infty), PM^2)$

$$\|B(u,v)\|_{L^\infty([0,\infty), PM^2)} \le \eta \|u\|_{L^\infty([0,\infty), PM^2)} \|v\|_{L^\infty([0,\infty), PM^2)}$$

Proof:

we estimate the Fourier transform of $B(u,v)$

$$|\xi|^2 \left| \int_0^t e^{-(t-s)|\xi|^2} i(a\circ\xi)(\hat u(\cdot,s) * \tilde v(\cdot,s))(\xi)\,ds \right| \quad ⊛$$

Step 1: $|\hat u(\cdot,s) * \hat v(\cdot,s)(\xi)| = |\int \hat u(\xi - z, s)\hat v(z,s)\,ds| \le$

$$\le \int_{\mathbb{R}^3} \frac{1}{|\xi - z|^2} \frac{1}{|z|^2}\,dz\, \|u(\cdot,s)\|_2 \cdot \|v(\cdot,s)\|_2 = (*)$$

It is easy to show that

$$\int_{\mathbb{R}^2} \frac{1}{|\xi - z|^2} \cdot \frac{1}{|z|^2}\,dz = \frac{c}{|\xi|}$$ (excersize + hint - use the Fourier transform)

so

$$(*) \le \frac{c}{|\xi|} \|u(\cdot,s)\|_2 \cdot \|v(\cdot,s)\|_2$$

Step 2:

$$⊛ \le |\xi|^2 |a| |\xi| \int_0^t e^{-(t-s)|\xi|^2}\,ds \cdot \frac{c}{|\xi|} \cdot \|u\|_2 \cdot \|v\|_2 = |a| \cdot c\,|\xi|^2 \int_0^t e^{-(t-s)|\xi|^2}\,ds \sup_{s>0} \|u(s)\|_2 \|v(s)\|_2$$

$$\le \eta \|u\|_{L^\infty([0,\infty), PM^2)} \|v\|_{L^\infty([0,\infty), PM^2)}$$

Lemma (Abstract existance theorem)

Let $(X, \|\cdot\|_X)$ be a Banach space, let $B: X\times X \to X$ be a bilinear form s.t. $\|B(u,v)\|_X \le \eta \|u\|_X \|v\|_X$

if $0 < \varepsilon < \frac{1}{4\eta}$ and if $\|y\|_X < \varepsilon$ then the equation

$u = y + B(u,u)$ has a solution in X

Moreover this is the unique solution satisfying $\|u\| \leq 2\varepsilon$

Excersize:

quadratic equation for u

$u = y + \eta u^2$

Find all y and η such that this equation has a solution

$\eta > 0$

$\Delta = 1 - 4\eta y \geq 0$

$y < \frac{1}{4\eta}$

we apply this abstract lemme to the "quadratic equation"

$$u(t) = \underbrace{S(t)u_0 + \int_0^t S(t-s)\,F(s)\,ds}_{\overset{\shortparallel}{y}} - \underbrace{\int_0^t S(t-s)(a\circ\nabla)\,u^2(s)\,ds}_{\overset{\shortparallel}{B(u,u)}}$$

in the space $L^\infty([0,\infty), PM^2)$

Theorem: Assume that $0 < \varepsilon < \frac{1}{4\eta}$, η is a constant from lemma,

$\forall\, u_0 \in PM^2$ $\forall F \subset L^\infty((0,\infty), PM^0)$ satisfying $\|u_0\|_2 + \|F\|_{L^\infty((0,\infty), PM^0)} < \varepsilon$

the problem has a solution in $X = L^\infty((0,\infty), PM^2)$

Moreover this is a unique solution satisfying

$\|u\|_X < 2\varepsilon$ (MAIN THEOREM TODAY)

self-similar solutions

$$\begin{cases} u_t - \Delta u + (a\circ\nabla)u^2 = F \\ u(x,0) = u_0(x) \end{cases}$$

Assume that $\forall \lambda > 0$

- $\lambda u_0(\lambda x) = u_0(x)$ (example : $u_0(x) = \frac{1}{|x|} \in PM^2$, $\hat{u}_0(\xi) = \frac{c}{|\xi|^2}$)

- $\lambda^3 F(\lambda x, \lambda^2 t) = F(x,t)$

examples : $F \equiv 0$, $F(x,t) = \frac{1}{t^{3/2}} H\left(\frac{x}{\sqrt{t}}\right)$

$F = \delta_0$ definirition if $F \in S'(\mathbb{R}^3)$ $(F(\lambda\cdot)\varphi) = F(\lambda^{-3}\varphi(\frac{\cdot}{\lambda}))$

$$(\lambda^3 \delta_0(\lambda\ )\ \varphi) = \lambda^3\lambda^{-3}\delta_0\varphi(\tfrac{\cdot}{\lambda}) = \varphi(0) = \delta\varphi$$

Lemma:

Under these assumptions on u_0 and F if $u = u(x,t)$ is a solution then $u_\lambda = \lambda u(\lambda x, \lambda^2 t)$ is also a solution

Corollary of the Main Theorem:

If, moreover, u_0, F in the main theorem are scaling invariant as above, then the corresponding solution satisfies

$u(x,t) = \lambda u(\lambda x, \lambda^2 t)$

(so u is self-similar)

stationary solutions:

Theorem 2: Assume that $u \in PM^2$, and $F \in PM^0$ ~~are~~ is independent of t.

Then the following facts are equivalent

1° $u = u(x)$ is a stationary solution of (CD) equation

which means that

$$u = S(t)u - \int_0^t S(t-s)(a\cdot\nabla)u^2 ds + \int_0^t S(t-s)F ds \qquad \forall\, t>0$$

2° u satisfies the integral equation:

$$u = - \int_0^\infty S(s)(a\cdot\nabla)u^2 ds + \int_0^\infty S(s) F ds$$

Corollary (of the main theorem):

For every $F \in PM^0$, $\|F\|_{PM^0} < \varepsilon$ we have a stationary solution of our equation.

Instead of proof of theorem 2:

$$\begin{cases} u_t = \Delta u + F(x,t) \\ u(x,0) = u_0(x) \end{cases}$$

solution (in our sense)

$$\hat{u}(\xi,t) = e^{-t|\xi|^2}\hat{u}_0(\xi) + \int_0^t e^{-(t-s)|\xi|^2} \hat{F}(\xi,s) ds$$

Now assume that F is independent of t and u is stationary $u(x,t) = u(x)$

$$0 = \Delta u + F \qquad \text{solution} \qquad \hat{u}(\xi) = \frac{1}{|\xi|^2}\hat{F}(\xi) = \int_0^\infty e^{-s|\xi|^2}\hat{F}(\xi) ds \iff$$

$$u = \int_0^\infty S(s) F ds$$

20.05.2011

Linear heat equation (one dimensional)

$$\begin{cases} u_t = u_{xx} \qquad x \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases}$$

$u(x,t) = G(t) * u_0$ - solution

$$G(x,t) = (4\pi t)^{-1/2} e^{-|x|^2/4t}$$

We have following theorem

$$\lambda(u(\lambda x, \lambda^2 t)) \xrightarrow{\lambda\to\infty} M\, G(x,t)$$

Diffrent point of view

Let $t = 1$. we want

$$\lambda u(\lambda x, \lambda^2) \xrightarrow[\lambda\to\infty]{} M\, G(x,1)$$

Denote $\lambda = t$

$$t u(tx, t^2) \xrightarrow[t\to\infty]{} M\, G(x,1)$$

Let $v(x,t) := t u(tx, t^2)$

Strategy: write the equation for v and study it's properties

Precise statement

Let $u = u(x,t)$ be a solution of $\begin{cases} u_t = u_{xx} \\ u(x,0) = u_0(x) \end{cases}$

we define the self - similar change of variables

$$v(x,t) = e^t u(xe^t, 2(e^{2t} - 1))$$

A direct calculation shows that

$$\begin{cases} v_t = v_{xx} + (xv)_x \\ v(x,0) = v_0(x) = u_0(x) \end{cases}$$

we study the behaviour of $v(x,t)$ as $t \to \infty$

Properties of solution :

- if $v_0(x) \geqslant 0$, then $v(x,t) \geqslant 0$
- $\int_{\mathbb{R}} v(x,t)\,dx = \int_{\mathbb{R}} v_0(x)\,dx \quad \forall\, t > 0$

proof

$$\int_{\mathbb{R}} v(x,t)\,dx = \int_{\mathbb{R}} e^t u(\underset{y}{\underset{\parallel}{xe^t}}, 2(e^{2t}-1))\,dx = \int_{\mathbb{R}} u(y, 2(e^{2t}-1))\,dy = \int_{\mathbb{R}} u_0(y)\,dy$$

we always assume that $v_0(x) \geqslant 0$

$$\int_{\mathbb{R}} v_0(x)\,dx = 1$$

Stationary solutions

we solve

$$0 = v_{xx} + (xv)_x$$

integrating

$$v_x + xv = c \quad / e^{x^2/2}$$

$$(ve^{x^2/2})_x = ce^{x^2/2} \quad / \int_0^x$$

$$v(x) = v(0)e^{-x^2/2} + Ce^{-x^2/2} \int_0^x e^{y^2/2}\,dy$$

we require $v \in L^1(\mathbb{R})$

Hence $c = 0$

$$v(x) = v(0)e^{-x^2/2}$$

we want $\int_{\mathbb{R}} v(x)\,dx = 1$

$$\text{so} \quad v(0) = \left(\int_{\mathbb{R}} e^{-x^2/2}\right)^{-1} = \frac{1}{\sqrt{2\pi}}$$

we proved

Theorem: The equation $v_t = v_{xxt} (xv)_x$ has a stationary solution

$$v_\infty(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = G(x, \tfrac{1}{2})$$

This is the unique stationary solution such that

$v_\infty(x) \geqslant 0$ and $\int_{\mathbb{R}} v_\infty(x)\,dx = 1$

Theorem: (Main results during this lecture)

For every $v_0 \in L^1(\mathbb{R})$, $v_0 \geq 0$ $\int_{\mathbb{R}} v_0 = 1$ the solution of problem

$$\begin{cases} v_t = v_{xx} + (xv)_x \\ v(x,0) = v_0(x) \end{cases}$$

satisfies

$$\| v(\cdot,t) - v_\infty(\cdot) \|_{L^1(\mathbb{R})} \leq Ce^{-\mu t} \quad \text{for all } t>0$$

for some constant $C>0$ and $\mu > 0$

Relative entropy method (Toscani 1996, Carrillo - Toscani 2000, Arnold-Toscani 2001)

Def: Let $\psi \in C([0,\infty)) \cap C^4((0,\infty))$, $\psi(1) = 0$, ψ - convex $\psi'' \geq 0$, $\psi'' \not\equiv 0$

For even $f, g \in L^1(\mathbb{R})$, $f \geq 0$, $g \geq 0$ $\int f = \int g = 1$

we define the relative entropy of f with respect to g:

$$\Sigma(f,g) = \int_{\mathbb{R}} \psi\left(\frac{f}{g}\right) \cdot g \, dx$$

Favourite examples:

$\psi(s) = (s-1)^2$

$\psi(s) = s \log s$ $\quad \psi'(s) = \log s + 1$

$\psi''(s) = \frac{1}{s} > 0$

Remark: For every $f \geq 0$, $g \geq 0$, $\int f = \int g = 1$

$\Sigma(f,g) \geq 0$

Here we apply the Jensen inequality

$$0 = \psi(1) = \psi\left(\int_{\mathbb{R}} f \, dx\right) = \psi\left(\int_{\mathbb{R}} \frac{f}{g} \cdot g \, dx\right) \overset{\text{Jen.}}{\leq} \int_{\mathbb{R}} \psi\left(\frac{f}{g}\right) \cdot g \, dx$$

Lemma: Let $v = v(x,t)$ be a solution with the initial data

$v_0 \in L^1(\mathbb{R})$, $v_0 \geq 0$, $\int v_0 = 1$

$$\frac{d}{dt} \Sigma(v(t), v_\infty) \leq 0$$

Proof:

we use the equation

$$v_t = (v_x + xv)_x = \left(v_\infty \left(\frac{v}{v_\infty}\right)_x\right)_x$$

we calculate

$$\frac{d}{dt} \Sigma(v(t), v_\infty) = \frac{d}{dt} \int \psi\left(\frac{v(x,t)}{v_\infty(x)}\right) v_\infty(x) \, dx = \int v_t \, \psi'\left(\frac{v}{v_\infty}\right) dx =$$

$$= \int \left(v_\infty \left(\frac{v}{v_\infty}\right)_x\right)_x \psi'\left(\frac{v}{v_\infty}\right) dx \overset{\text{by parts}}{=} \underbrace{- \int_{\mathbb{R}} \left[\left(\frac{v}{v_\infty}\right)_x\right]^2 \psi''\left(\frac{v}{v_\infty}\right) v_\infty \, dx}_{I(t)} \leq 0$$

Lemma 2:

Let $I(t) = - \int_{\mathbb{R}} \left[\left(\frac{v}{v_0}\right)_x\right]^2 \psi''\left(\frac{v}{v_\infty}\right) \cdot v_\infty \, dx$

If ψ satisfies moreover $(\psi''')^2 \leq \frac{1}{2} \psi'' \psi^{(iv)}$

then

$$\frac{d}{dt} I(t) \geq -2 I(t)$$

Proof: 2 pages of calculations

Conclusion:

$$\frac{d}{dt}\Sigma(v(t), v_\infty) = I(t) \leq -\frac{1}{2}\frac{d}{dt} I(t)$$

$$\frac{d}{dt} I(t) \geq -2\frac{d}{dt}\Sigma(v(t), v_\infty) \quad \Big| \int_t^\infty ds$$

suppose for a moment, that $I(t) \to 0$ and $\Sigma(v(t), v_\infty) \xrightarrow{t\to\infty} 0$

$$-I(t) \geq 2\Sigma(v(t), v_\infty)$$

$$I(t) \leq -2\Sigma(v(t), v_\infty)$$

we integrate the inequality

$$\Sigma(v(t), v_\infty) < \Sigma(v_0, v_\infty)e^{-2t}$$

Remark 1:

$I(t) \to 0$ when $t \to \infty$, ~~because~~ then

- $I(t) \leq 0$
- $\frac{d}{dt} I(t) \geq -2I(t)$

Excersize: cheak what we 8 can get from it

Remark 2:

$\Sigma(v(t), v_\infty) \to 0$

it will be explained later

we proved the inequality

$$\int_{\mathbb{R}} \psi\left(\frac{v(t)}{v_\infty}\right) v_\infty \, dx \leq \int_{\mathbb{R}} \psi\left(\frac{v_0}{v_\infty}\right) v_\infty \, dx \, e^{-2t}$$

Csiszar - Kullback inequality

$$\int_{\mathbb{R}} \psi\left(\frac{f}{g}\right)\cdot g \, dx \geq K \|f-g\|^2_{L^1(\mathbb{R})}$$

for every $f, g \geq 0$, $\int f = \int g = 1$

Proof of C-K inequality:

Assume that $f(x) \geq g(x)$

Let

$$h(x) = \frac{f(x)-g(x)}{g(x)} \quad \text{then} \quad \frac{f(x)}{g(x)} = 1+h(x)$$

Taylor

$$\psi\left(\frac{f}{g}\right) = \psi(1+h) \overset{\text{Taylor}}{=} \psi(1) + \psi'(1)h + \frac{1}{2}\psi''(\theta)h^2$$

$$\int \psi\left(\frac{f}{g}\right) g \, dx = \psi'(1)\int h\cdot g \, dx + \frac{1}{2}\int \psi''(\theta)h^2 \cdot g$$

$$\int_{\mathbb{R}} \psi\left(\frac{f}{g}\right)\cdot g \, dx = \frac{1}{2}\int \psi''(\theta)\frac{(f(x)-g(x))^2}{g(x)} dx \overset{\text{schwartz}}{\geq} K\|f-g\|_1^2 \quad \text{because} \quad \int g = 1$$

$K = \inf \psi''$

This completes the proof of the main theorem

Important:

$$I(t) \leq -2\Sigma(v(t), v_\infty)$$

this is a family of the sobolev type inequality

$$\int \psi\left(\frac{v}{v_\infty}\right) v_\infty \leq \frac{1}{2} \int \left[\sqrt{\psi''\left(\frac{v}{v_\infty}\right)} \left(\frac{v}{v_\infty}\right)_x \right]^2 dx$$

Denoting

$$F(x) = \int_1^x \sqrt{\psi''(s)}\, ds$$

Then we have

$$\int \psi\left(\frac{v}{v_\infty}\right) v_\infty \leq \frac{1}{2} \int \left[\frac{d}{dx} F\left(\frac{v}{v_\infty}\right)\right]^2 v_\infty \, dx$$

$$\int_{\mathbb{R}} \psi(w)\, v_\infty(x)\, dx \leq \int_{\mathbb{R}} \left[\frac{d}{dx} F(w)\right]^2 v_\infty(x)\, dx$$

For example
$\psi(s) = s\log s$
$\psi''(s) = \frac{1}{s}$
$F(s) = 2\sqrt{s} - 2$

$$\int_{\mathbb{R}} v \cdot \log \frac{v}{v_\infty}\, dx \leq \frac{1}{2} \int \left[2 \frac{d}{dx} \sqrt{\frac{v}{v_\infty}} \right]^2 v_\infty\, dx$$

Logarithmic Sobolev inequality

Final example

Porous medium equation

$$\begin{cases} v_t = \Delta v^m & x \in \mathbb{R}^n, \ t > 0 \\ v(x,0) = v_0(x) & m > 1 \quad \text{fixed parameter} \end{cases}$$

1° self - similar change of variable

Define $u(x,t) = e^{nt}\, v(x e^{t}, k(e^{\frac{t}{k}} - 1))$ $\qquad k = \frac{1}{n(m-1)+2}$

The new function satisfies

$$u_t = \Delta u^m + \nabla(xu)$$
$$u(x,0) = u_0(x)$$

2° stationary solutions

$$\Delta u^m + \nabla(xu) = 0$$
$$\nabla(\nabla u^m + xu) = 0$$

stationary solution

$$u_\infty(x) = \left(C - \frac{m-1}{2m}|x|^2\right)_+^{\frac{1}{m-1}}$$

3° Relative entropy

$$\Sigma(u(t)) = \int_{R^n} \frac{u(x,t)^m\, u(x,t)}{m-1} + \frac{1}{2}|x|^2 u(x,t)\, dx$$

$$\Sigma(u(t), u_0) = \Sigma(u(t)) - \Sigma(u_\infty)$$

4° $\frac{d}{dt} \Sigma(u(t), u_\infty) \leq - C\, \Sigma(u(t), u_\infty)$

5° Csiszar - Kullback

Nice example at the end

$$\begin{cases} u_t = \Delta u & x \in \Omega \quad , \quad \Omega \subseteq \mathbb{R}^n \text{ bounded} \\ u(x,0) = u_0(x) & \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

1° Stationary solution

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \partial\Omega \end{cases} \Rightarrow u \equiv 0$$

2° "Relative entropy"

$$\Sigma(u(t)) = \int_\Omega u^2(x,t)\,dx$$

$$\frac{d}{dt}\Sigma(u(t)) = 2\int_\Omega u_t\, u\,dx = 2\int_\Omega \Delta u\, u\,dx = -2\int |Du|^2 dx \overset{\text{Poincare}}{\leq} -2\mu \int u^2 dx$$

this implies

$$\Sigma(u(t)) \leq \Sigma(u_0)\, e^{-2\mu t} = -2\mu\, \Sigma(u(t))$$