15. XII. 2011

Ex (from last classes)

Give an example of measures $\mu$ and $\nu$ such that $\nu$ cannot be represented as $\nu_{ac} + \nu_s$, where $\nu_{ac} << \mu$ and $\nu_s \perp \mu$.

last time we made a mistake ( it wasn't a contradiction ).

Assume that $\nu = \nu_{ac} + \nu_s$ such that $\nu_{ac} << \mu$, $\nu_s \perp \mu$

From definition : There exists a set $D$ such that

$\mu(D) = \nu_s(D^c) = 0$

Let us take $\begin{cases} \mu - \text{Lebesgue measure} \\ \nu - \text{c\"onting measure} \end{cases}$

Then we can chose $A \subset D^c$ s.t. $A$ is countable

$A \subset D^c \Rightarrow \nu_s(A) = 0$

From countability of $A$ we have $\mu(A) = 0$

$\nu_{ac} << \mu \Rightarrow \nu_{ac}(A) = 0$

$\nu(A) = \nu_{ac} + \nu_s(A) = 0$

we choose $A \neq 0$ $\lightning$

Ex 1. Let $\mu, \mu_k$ be Radon measures on $\mathbb{R}^n$. Prove that the following statements are equivalent :

a) $\lim\limits_{k\to\infty} \int_{\mathbb{R}^n} f d\mu_k = \int_{\mathbb{R}} f d\mu \quad \forall f \in C_0(\mathbb{R}^n)$

b) $\limsup\limits_{k\to\infty} \mu_k(K) \leq \mu(K)$ for any compact $K$ and

$\liminf\limits_{k\to\infty} \mu_k(U) \geq \mu(U)$ for any open $U$.

c) $\lim\limits_{k\to\infty} \mu_k(B) = \mu(B)$ for each bounded Borel set such that $\mu(\partial B) = 0$.

Proof:

a) $\Rightarrow$ b)

~~For every f~~ $f = \mathbb{1}_K * g_\varepsilon$

$\mathbb{1}_K \leq f \leq \mathbb{1}_{K_\varepsilon}$

$\mathbb{R}^n \setminus K^\varepsilon \cap K = \emptyset$

$\hookrightarrow$ from Urysohn's lemma $\exists f$ s.t. $f|_{\mathbb{R}^n \setminus K^\varepsilon} = 0$, $f|_K = 1$

for Radon measure's we have $\mu(K) = \inf\{\mu(K^\varepsilon) : K \subset K^\varepsilon\}$

$$\mu_k(K) \le \int_{\mathbb{R}^n} f\,d\mu_k \xrightarrow{k\to\infty} \int_{\mathbb{R}} f\,d\mu \le \mu(K^\varepsilon) \qquad \forall_\varepsilon$$

$$\overline{\lim}\, \mu_k(K) \le \mu(K^\varepsilon) \qquad \forall_\varepsilon$$

$$\overline{\lim}\, \mu_k(K) \le \mu(K)$$

Similarly we chose

$$\mathbb{1}_{U_\varepsilon} \le f \le \mathbb{1}_U$$

$$\int f\,d\mu_k \le \mu_k(U)$$

Again for Radon measure we have $\mu(A) = \sup\{\mu(K) \mid K \subset A,\ K\text{-compact}\}$ for every $A$ - $\mu$-m.

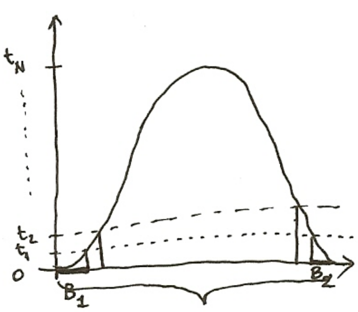
$$\lim_{k\to\infty} \int f\,d\mu_k \le \liminf_{k\to\infty} \mu_k(U)$$

$$\mu(U) \le \liminf_{k\to\infty} \mu_k(U)$$

b) $\Rightarrow$ c)

$$\mu(B) = \mu(B^\circ) \le \liminf \mu_k(B^\circ) \le \overline{\lim}\, \mu_k(\bar{B}) \le \mu(\bar{B}) \le \underset{0}{\underset{\shortparallel}{\mu(\partial B)}} + \mu(B)$$

c) $\Rightarrow$ a) Take $f \in C_0(\mathbb{R}^n)$ (assume for the moment that this is nonnegative)




$B(0,R)$

There exists $R > 0$ such that $\operatorname{supp} f \subset B(0,R)$ and $\mu(\partial B(0,R)) = 0$

$$f^{-1}((t_{i-1}, t_i]) = B_{i-1} \leftarrow \text{borel}$$

$$\mu(f^{-1}(\{t_i\})) = 0$$

$$\sum_{i=1}^N \mu(B_i)\cdot t_{i-1} \le \int_{\mathbb{R}} f\,d\mu \le \sum_{i=1}^N \mu(B_i)\, t_i$$

$$\sum_{i=1}^N \mu_k(B_i)\cdot t_{i-1} \le \int_{\mathbb{R}} f\,d\mu_k \le \sum_{i=1}^N \mu_k(B_i)\cdot t_i$$

$k\to\infty$ $\downarrow$ from construction of those sets

$$\sum_{i=1}^N \mu(B_i)\cdot t_{i-1} \le \lim_{k\to\infty} \int_{\mathbb{R}} f\,d\mu_k \le \sum_{i=1}^N \mu(B_i)\cdot t_i \qquad \text{for every } N$$

Passing with N to the limit we get

$$\int f d\mu \le \lim_{k\to\infty} \int f d\mu_k \le \int f d\mu$$ ∎

Approximate limit:

Def: Let $f: \mathbb{R}^n \to \mathbb{R}^m$. We say that $g$ is the approximate limit of $f$ as $y \to x$ if for every $\varepsilon > 0$

$$\lim_{r\to 0} \frac{\lambda^n(B(x,r)\cap A)}{\lambda^n(B(x,r))} = 0 \qquad A = \{y : |f(y) - g| \geq \varepsilon\}$$

we write $\operatorname{ap}\lim_{y\to x} f(y) = g$

HOMEWORK:

1) Prove that approximate limit is unique

2) Show that every measurable function is approximately continuous $(f: \mathbb{R}^m \to \mathbb{R}^n)$