Measure theory

5. X. 2011

Literature:

Evans and G. „Measure theory and fine properties of the functions"

- first two chapters

Federer „Geometric measure theory"

Let $X$ be a set and $2^X$ - set of all subsets.

Def:
function $\mu : 2^X \to [0, +\infty)$ is measure on $X$ iff:

(i) $\mu(\emptyset) = 0$

(ii) $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ for $A \subset \bigcup_{k=1}^{\infty} A_k$

Claim:
If $\mu$ - measure on $X$ and $A \subset B \subset X$ then $\mu(A) \leq \mu(B)$

Def:
Let $\mu$ - measure on $X$ and $A \subset X$ then restriction of the measure $\mu$ to $A$ is defined as

$$(\mu \llcorner A)(B) = \mu(A \cap B) \quad \forall B \subset X$$

Def:
Let $A \subset X$ then $A$ is $\mu$-measurable iff:

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) \quad \forall B \subset X$$

Remarks:

1) If $\mu(A) = 0$ then $A$ is $\mu$-measurable
2) $A$ $\mu$-measurable $\Leftrightarrow$ $X \setminus A$ is $\mu$-measurable
3) If $A \subset X$ and $B$ is $\mu$-measurable $\Rightarrow$ $B$ is $\mu \llcorner A$-measurable

Theorem:
Let $\{A_k\}_{k=1}^{\infty}$ sequence of $\mu$-measurable sets then:

(i) sets $\bigcup_{k=1}^{\infty} A_k$ and $\bigcap_{k=1}^{\infty} A_k$ are $\mu$-measurable

(ii) If sets $\{A_k\}_{k=1}^{\infty}$ s.t. $A_k \cap A_j = \emptyset$ for $j \neq k$ then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

(iii) If $A_1 \subset \ldots \subset A_k \subset \ldots$ then $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} \mu(A_k)$

(iv) If $A_1 \supset \ldots \supset A_k \supset A_{k+1} \supset \ldots$ and $\mu(A_1) < +\infty$ then

$$\lim_{k \to +\infty} \mu(A_k) = \mu\left(\bigcap_{k=1}^{\infty} A_k\right)$$

proof:

If $A, B \subseteq X$ then

$$\mu(B) \leq \mu(B \cap A) + \mu(B \setminus A)$$

Let $B \subset X$ $A_1, A_2$ - $\mu$-measurable

$$\mu(B) = \mu(B \cap A_1) + \mu(B \setminus A_1) = \mu(B \cap A_1) + \mu((B \setminus A_1) \cap A_2) + \mu((B \setminus A_1) \setminus A_2)$$

but

$$\mu(B \cap A_1) \geq \mu(B \cap (A_1 \cup A_2)) - \mu((B \setminus A_1) \cap A_2)$$

and

$$((B \setminus A_1) \setminus A_2) = B \setminus (A_1 \cup A_2)$$

hence

$$\mu(B) \geq \mu(B \cap (A_1 \cup A_2)) + \mu(B \setminus (A_1 \cup A_2))$$

$A_1 \cup A_2$ $\mu$-measurable

$$X \setminus (A_1 \cap A_2) = (X \setminus A_1) \cup (X \setminus A_2)$$

$\Rightarrow$ $A_1 \cap A_2$ is $\mu$-measurable

$\Rightarrow$ (i)

(ii) Let $\{A_k\}_{k=1}^{\infty}$ $A_k \cap A_j = \emptyset$ $j \neq k$

$$B_j = \bigcup_{k=1}^{j} A_j$$

then

$$\mu(B_{j+1}) = \mu(B_{j+1} \cap A_{j+1}) + \mu(B_{j+1} \setminus A_{j+1}) = \mu(A_{j+1}) + \mu(B_j)$$

hence

$$\mu\left(\bigcup_{k=1}^{j} A_k\right) = \sum_{k=1}^{j} \mu(A_k) \qquad j = 1, \dots$$

Note that

$$\mu\left(\bigcup_{k=1}^{+\infty} A_k\right) \geq \lim_{j \to \infty} \mu\left(\bigcup_{k=1}^{j} A_k\right)$$

$$\sum_{k=1}^{\infty} \mu(A_k) \leq \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

(iii) $A_{k+1} \setminus A_k$

$$\lim_{k \to +\infty} \mu(A_k) = \mu(A_1) + \sum_{k=1}^{\infty} \mu(A_{k+1} \setminus A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

(iv) $$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) \leq \lim_{k \to +\infty} \mu(A_k)$$

$$\left(\bigcap_{i=1}^{\infty} A_i \subset A_k\right)$$

$$\mu(A_1) - \lim_{k \to +\infty} \mu(A_k) = \lim_{k \to \infty} \mu(A_1 \setminus A_k) \overset{(iii)}{=} \mu\left(\bigcup_{k=1}^{\infty} A_1 \setminus A_k\right)$$

Note that

$$A_1 = \bigcup_{k=1}^{\infty} (A_1 \setminus A_k) \cup \bigcap_{k=1}^{\infty} A_k$$

hence

$$\mu\left(\bigcup_{k=1}^{\infty} A_1 \setminus A_k\right) \geq \mu(A_1) - \mu\left(\bigcap_{k=1}^{\infty} A_k\right)$$

$$\lim_{k \to +\infty} \mu(A_k) \leq \mu\left(\bigcap_{k=1}^{+\infty} A_k\right)$$

$B_j = \bigcup_{k=1}^{j} A_k$ $\mu$- measurable $B$ that $\mu(B) < +\infty$

$$\mu\left(B \cap \bigcup_{k=1}^{\infty} A_k\right) + \mu\left(B \setminus \bigcup_{k=1}^{\infty} A_k\right) = (\mu L B)\left(\bigcup_{k=1}^{\infty} B_k\right) + (\mu L B)\left(\bigcap_{k=1}^{\infty} (X \setminus B_k)\right) =$$

$$= \lim_{k \to +\infty} (\mu L B)(B_k) + \lim_{k \to +\infty} (\mu L B)(X \setminus B_k) = \mu(B)$$

because

$$(\mu L B)(B_k) + (\mu L B)(X \setminus B_k) = \mu(B \cap B_k) + \mu(B \cap (X \setminus B_k)) =$$

$$= \mu(B \cap B_k) + \mu(B \setminus B_k) = \mu(B)$$

$$X \setminus \bigcap_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} (X \setminus A)$$

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**Def:** $\mathcal{A} \subset 2^X$ is a $\sigma$- algebra iff:

(i) $\emptyset, X \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow X \setminus A \in \mathcal{A}$

(iii) $A_k \in \mathcal{A}$ $(k = 1, 2, \dots) \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

**Claim:**
measurable sets are $\sigma$- algebra's

**Def:** set $A \subset X$ is $\sigma$- finite w.r.t. $\mu$ if

$A = \bigcup_{k=1}^{\infty} B_k$ $\mu(B_k) < \infty$ and $B_k$ $\mu$-m

**Def:** Borel $\sigma$-algebra is smollest $\sigma$- algebra with all open sets.

**Def:** (i) measure $\mu$ is regular on $X$ if $\forall A \subset X$ there exists set $B$ $\mu$-m s.t. $A \subset B$ and $\mu(A) = \mu(B)$

(ii) measure $\mu$ is Borel (on $\mathbb{R}^n$) if any Borel set is measurable

(iii) measure $\mu$ on $\mathbb{R}^n$ is Borel regular if it is Borel, moreover $\forall\ A \subset \mathbb{R}^n$ there exists $B$ - Borel s.t. $A \subset B$ and $\mu(A) = \mu(B)$

Note: Borel regular $\mu \neq$ regular Borel $\mu$ !

(iv) measure $\mu$ on $\mathbb{R}^n$ is Radon measure if it is Borel regular and $\mu(K) < \infty$ for all $K$ - compact

Theorem 1:

Let $\mu$ - regular measure on $X$ and $A_1 \subset A_2 \subset \dots \subset A_k \subset \dots$

then $\lim\limits_{N\to\infty} \mu(A_N) = \mu\left(\bigcup\limits_{k=1}^{\infty} A_k\right)$.

Proof:

Hence $\mu$ is regular then there exists sequence of $\mu$-m sets $C_k$ s.t. $A_k \subset C_k$ $\mu(A_k) = \mu(C_k)$

(in general $C_k \not\subset C_{k+1}$ )

Let $B_k = \bigcap\limits_{j\geq k} C_j$ then $B_k \subset B_{k+1}$

$B_k$ $\mu$-m, $A_k \subset B_k$, $\mu(A_k) \leq \mu(B_k) \leq \mu(C_k)$

$\lim\limits_{k\to\infty} \mu(A_k) = \lim\limits_{k\to\infty} \mu(B_k) = \mu\left(\bigcup\limits_{k=1}^{\infty} B_k\right) \geq \mu\left(\bigcup\limits_{k=1}^{\infty} A_k\right)$

$A_k \subset \bigcup\limits_{i=1}^{\infty} A_i$ $\lim\limits_{k\to\infty} \mu(A_k) \leq \mu\left(\bigcup\limits_{k=1}^{\infty} A_k\right)$ ∎

Theorem:

Let $\mu$ - Borel regular measure on $\mathbb{R}^n$. Let assume that $A \subset \mathbb{R}^n$ ($\mu$-m) and $\mu(A) < +\infty$ then $\mu \llcorner A$ is Radon.

Remark:

If we have not $\mu(A) < \infty$ then Borel regular

Proof:

1) $(\mu \llcorner A)(K) \leq +\infty$ $\forall\ K \subset \mathbb{R}^n$ s.t. $K$ - compact

2) ~~Hence~~ since arbitrary set $\mu$-m is $(\mu \llcorner A)$-m then $\mu \llcorner A$ - Borel

$(\mu \llcorner A)$ is Borel regular

$\mu$ is Borel regular and then there exists $B$ - Borel s.t.
$A \subset B$ and $\mu(A) = \mu(B)$ moreover
$\mu(B \setminus A) = \mu(B) - \mu(A) = 0$.

Let $C$ - arbitrary, $C \subset \mathbb{R}^n$ then

$$(\mu \llcorner B)(C) = \mu(C \cap B) = \mu(C \cap B \cap A) + \mu((C \cap B) \setminus A) \le \mu(C \cap A) + \underbrace{\mu(B \setminus A)}_{=0} = \le (\mu \llcorner A)(C)$$

Now we want to show
$\forall C \subset \mathbb{R}^n \; \exists D$ - Borel s.t. $(\mu \llcorner B)(C) = (\mu \llcorner B)(D)$

$B \cap C \subset E$ s.t. $\mu(E) = \mu(B \cap C) = (\mu \llcorner B)(C)$.
Let $D = E \cup (\mathbb{R}^n \setminus B)$ (Borel) and $C \subset (B \cap C) \cup (\mathbb{R}^n \setminus B) \subset \cancel{B}$
$\subset E \cup (\mathbb{R}^n \setminus B) = D$.

$$(\mu \llcorner B)(D) = \mu(D \cap B) = \mu(E \cap B) \le \mu(E) = \mu(B \cap C) = (\mu \llcorner B)(C)$$

Approximation by open and closed sets.

<u>Lemma:</u>
Let $\mu$ - Borel measure on $\mathbb{R}^n$ and $B$ Borel set then
(i) If $\mu(B) < +\infty$ then $\forall \varepsilon > 0$ there exists ~~closed~~ compact set $C_\varepsilon$ s.t. $C_\varepsilon \subset B$ and $\mu(B \setminus C_\varepsilon) < \varepsilon$.
(ii) If $\mu$ is Radon measure then $\forall \varepsilon > 0$ $\exists$ (open) $U_\varepsilon$ s.t. $B \subset U_\varepsilon$ and $\mu(U_\varepsilon \setminus B) < \varepsilon$.

<u>Proof:</u>
(i) Let $\nu = \mu \llcorner B$, $\mu$ - Borel and $\mu(B) < \infty$ then $\nu$ - finite Borel measure

Let $\mathcal{F} = \{ A \subset \mathbb{R}^n : A$ is $\mu$-m and $\forall \varepsilon > 0$ there exists compact set $C_\varepsilon$ s.t. $C_\varepsilon \subset A$ $\nu(A \setminus C_\varepsilon) < \varepsilon \}$

1) $\{A_i\}_{i=1}^\infty \subset \mathcal{F} \Rightarrow A = \bigcap_{i=1}^\infty A_i \in \mathcal{F}$
indeed:
$A_i \in \mathcal{F}$ then $\exists C_i \subset A_i$ s.t. $\nu(A_i \setminus C_i) < \frac{\varepsilon}{2^i}$

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Let $C_\varepsilon = \bigcap_{i=1}^\infty C_i$ - closed set

$$\nu(A \setminus C) = \nu\left(\bigcap_{i=1}^\infty A_i \setminus \bigcap_{i=1}^\infty C_i\right) \le \nu\left(\bigcup_{i=1}^\infty (A_i \setminus C_i)\right) \le \varepsilon$$

2) $\{A_i\}_{i=1}^\infty \subset \mathcal{F} \Rightarrow A = \bigcup_{i=1}^\infty A_i \subset \mathcal{F}$ indeed:

$$\lim_{m\to+\infty} \nu\left(A \setminus \bigcup_{i=1}^{m} C_i\right) = \nu\left(\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{\infty} C_i\right) \leq \nu\left(\bigcup_{i=1}^{\infty} (A_i \setminus C_i)\right)$$

$$\leq \sum_{i=1}^{\infty} \nu(A_i \setminus C_i) < \varepsilon.$$

3) $G = \{A \in \mathcal{F} : \mathbb{R}^n \setminus A \in \mathcal{F}\}$

open sets in separable space can be covered by closed sets $\Rightarrow$ we have both closed and open sets.

(ii) Let $U_m = B(0,m)$

$U_m \setminus B$ - borel set and hence $\mu$ - Radon then $\mu(U_m \setminus B) < +\infty$.

$C_m \, \forall m \subset U_m \setminus B$ s.t. $\mu((U_m \setminus C_m) \setminus B_m) = \mu((U_m \setminus B) \setminus C_m) \leq \frac{\varepsilon}{2^m}$.

Let $U = \bigcup_{m=1}^{\infty} U_m \setminus C_m$ then $U$ - open. Hence $B \subset \mathbb{R}^n \setminus C_m$ then

$U_m \cap B \subset U_m \setminus C_m$ and $B = \bigcup_{m=1}^{\infty} (U_m \cap B) \subset \bigcup_{m=1}^{\infty} (U_m \setminus C_m) = U$

$$\mu(U \setminus B) = \mu\left(\bigcup_{m=1}^{\infty} ((U_m \setminus C_m) \setminus B)\right) \leq \sum_{m=1}^{\infty} \mu((U_m \setminus C_m) \setminus B) < \varepsilon$$

$\blacksquare$

**Theorem:**

Let $\mu$ Radon measure on $\mathbb{R}^n$ then

(i) $\forall A \subset \mathbb{R}^n$ $\mu(A) = \inf\{\mu(U) \mid A \subset U,\ U \text{ - open}\}$

(ii) $\forall A \subset \mathbb{R}^n$ $\mu$-m $\mu(A) = \sup\{\mu(K) \mid K \subset A,\ K \text{ - compact}\}$

**Proof:**

(i) $\mu(A) < +\infty$. Let assume that $A$ is Borel

$U_\varepsilon \supset A$ and $\mu(U_\varepsilon \setminus A) < \varepsilon$.

$$\inf_{\varepsilon > 0} \mu(U_\varepsilon) \underset{=}{\leq} \mu(A)$$

(ii) Let $A$ $\mu$-m s.t. $\mu(A) < +\infty$ and define $\nu = \mu \llcorner A$

($\nu$ - is Radon)

Let $\varepsilon > 0$ applying point (i) to the set $\mathbb{R}^n \setminus A$ we get that there exists $U$- open s.t. $\mathbb{R}^n \setminus A \subset U$ $\nu(U) < \varepsilon$

Let $C = \mathbb{R}^n \setminus U$ ( $C$- closed and $C \subset A$ ) then

$\mu(A \setminus C) = \nu(\mathbb{R}^n \setminus C) = \nu(U) < \varepsilon$

$0 \leq \mu(A) - \mu(C) < \varepsilon$

and

$\mu(A) = \sup\{\mu(C) : C \subset A \quad C \text{- closed}\}$

Let assume $\mu(A) = +\infty$

Let define $D_k = \{ x \mid k-1 \le |x| < k \}$ $\sum_{k=1}^{\infty} D_k = \mathbb{R}^n$

$\infty = \mu(A) = \sum_{k=1}^{\infty} \mu(A \cap D_k)$ but

$\mu(D_k \cap A) < +\infty$. Therefore by (i) there exist

closed sets $C_k \subset D_k \cap A_k$ s.t. $\mu(C_k) \geq \mu(D_k \cap A_k) - (\frac{1}{2})^k$

$\bigcup_{k=1}^{\infty} C_k \subset A$ and

~~lim~~

$\lim_{n\to\infty} \mu(\bigcup_{k=1}^{\infty} C_k) = \mu(\bigcup_{k=1}^{\infty} C_k) = \sum_{k=1}^{\infty} \mu(C_k) \geq$

$$\geq \sum_{k=1}^{\infty} [\mu(D_k \cap A) - (\tfrac{1}{2})^k] = +\infty$$

Theorem:

Let $\mu$ - measure on $\mathbb{R}^n$. If $\mu(A \cup B) = \mu(A) + \mu(B)$ for all sets $A, B$ s.t. $\operatorname{dist}(A,B) > 0$, then $\mu$ - Borel measure.

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$X, Y$ - topological spaces, $\mu$ - measure on $X$

Def: $f: X \to Y$ is $\mu$-m if for every $U$ - open $(U \subset Y)$ $f^{-1}(U)$ is $\mu$-m

Remark: $f: X \to Y$ is $\mu$-m then $f^{-1}(B)$, $B$ - borel is also $\mu$-m

Def: $f$ is Borel if $f^{-1}(U)$ - borel set

Def: We say that $f: X \to [-\infty, +\infty]$ is $\sigma$-finite w.r.t. $\mu$ if $f$ is $\mu$-m and $\{x \mid f(x) \neq 0\}$ is $\sigma$-finite.

Theorem:

(i) $f, g: X \to \mathbb{R}$ $\mu$-m $\Rightarrow$ $f+g$, $f \cdot g$, $|f|$, $\min\{f,g\}$, $\max\{f,g\}$ are $\mu$-m.

Moreover if $g \neq 0$ on $X$ then $f/g$ - $\mu$-m.

(ii) $f_k: X \to [-\infty, +\infty]$ $f_k$ - $\mu$-m then

$\inf_{k\in\mathbb{N}} f_k$, $\sup_{k\in\mathbb{N}} f_k$, $\liminf_{k\to\infty} f_k$, $\limsup_{k\to\infty} f_k$ is $\mu$-m

Sketch of the proof:

Idea - topology on $\overline{\mathbb{R}}$ is generated by $[-\infty, a]$, $[-\infty, a)$, $[a, +\infty]$, $(a, +\infty]$.

$(f+g)^{-1}([-\infty, a)) = \bigcup_{\substack{s,r \in \mathbb{R} \\ s+r<a}} (f^{-1}[-\infty, r) \cap g^{-1}([-\infty, s)) = \bigcup_{\substack{s,r \in \mathbb{Q} \\ s+r<a}} (f^{-1}[-\infty, r) \cup g^{-1}([-\infty, s))$

$(f \cdot g)^{-1}($

$(f^2)^{-1}((-\infty, a]) = f^{-1}([-\infty, a^{1/2}]) \setminus f^{-1}([-\infty, -a^{1/2}])$

$f \cdot g = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$

$\frac{1}{g}$

$$\left(\frac{1}{g}\right)^{-1}([-\infty, a)) = \begin{cases} g^{-1}((\frac{1}{a}, 0)) & \text{for } a < 0 \\ g^{-1}((-\infty, 0)) & a = 1 \\ g^{-1}((-\infty, 0) \cup g^{-1}((\frac{1}{a}, +\infty)) & a > 0 \end{cases}$$

$\frac{f}{g}$ $\quad f^{+} = \max\{f, 0\} = f \cdot \chi_{\{f \geq 0\}}$

$f^{-} = \max\{-f, 0\} = f \cdot \chi_{\{f < 0\}}$

$|f| = f^{+} - f^{-}$ $\quad \max\{f, g\} = (f-g)^{+}$

$\min\{f, g\} = -(f-g)^{-}$

$$\left(\inf_{k \in \mathbb{N}} f_k\right)^{-1}([-\infty, a)) = \bigcup_{k=0}^{\infty} f^{-1}([-\infty, a))$$

$$\left(\sup_{k \in \mathbb{N}} f_k\right)^{-1}([-\infty, a)) = \bigcap_{k=1}^{\infty} f^{-1}([-\infty, a))$$

$$\liminf_{k \to +\infty} f_k = \sup_{m \geq 1} \inf_{k \geq m} f_k$$

$$\limsup_{k \to +\infty} f_k = \inf_{m \geq 1} \sup_{k \geq m} f_k$$

Theorem: $f: X \to [0, +\infty]$ $\mu$-m then there exist family of sets $\{A_k\}_{k \in \mathbb{N}} \subset X$ s.t. $f = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}$

Remarks:

1) Any continuous function is borel

2) $f$ borel, $g$ - $\mu$-m $\Rightarrow$ $(f(g(\cdot))$ is $\mu$-m.

3) $f, g: X \to Y$, $Y$-linear topological space then

$f+g$ is also $\mu$-m

$\lambda \cdot f$ is $\mu$-m

Theorem:

Let $K \subset \mathbb{R}^n$ compact and $f: K \to \mathbb{R}^m$ continuous, then there exists continuous function $\bar{f}: \mathbb{R}^n \to \mathbb{R}^m$ s.t. $f(x) = \bar{f}(x)$ $\forall x \in K$

Proof:

$\mathcal{U} = \mathbb{R}^n \setminus K$ and $x \in \mathcal{U}$, $s \in K$, we define

$$u_s(x) = \max\left\{2 - \frac{|x-s|}{\operatorname{dist}(x, K)}, 0\right\}$$

- continuous function

$0 \leq u_s(x) \leq 1$

- $\sigma(x) = \sum_{j=1}^{\infty} 2^{-j} u_{s_j}(x)$ ; $0 \leq \sigma(x) \leq 1$

$$v_k(x) = \frac{2^{-k} u_{s_k}(x)}{\sigma(x)}$$

$$\sum_{k=1}^{\infty} v_k = 1$$

Define
$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in K \\ \sum_{k=1}^{\infty} v_k(x) f(s_k) & \text{if } x \in \mathcal{U} \end{cases}$$

$\bar{f}$ is of course continuous on $K$

$\bar{f}$ is continuous on $\mathcal{U}$, indeed

$$\sum_{k=l}^{\infty} | v_k(x) \cdot f(s_k) | \leq \left( \sum_{k=l}^{\infty} |v_k(x)| \right) \cdot \|f\|_\infty$$

$$\lim_{\substack{x \to a \in K \\ x \in \mathcal{U}}} f(x) = f(a)$$

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $|f(a) - f(s_k)| < \varepsilon$ if $|a - s_k| < \delta$

Let $x \in \mathcal{U}$ s.t. $|x - a| < \frac{\delta}{4}$

If $|a - s_k| \geq \delta$ then $\delta \leq |a - s_k| \leq |a - x| + |x - s_k| < \frac{\delta}{4} + |x - s_k|$

and

$$|x - s_k| \geq \frac{3}{4}\delta > 2|x-a| \geq 2 \operatorname{dist}(x, K)$$

and

$v_k(x) = 0$ if $|x-a| < \frac{\delta}{4}$ and $|a - s_k| \geq \delta$

$v_k(x) \geq 0$ and $\sum_{k=1}^{\infty} v_k(x) = 1$ for all $x \in \mathcal{U}$ s.t. $|x-a| < \frac{\delta}{4}$

$$|\bar{f}(x) - f(a)| \leq \sum_{k=1}^{+\infty} v_k(x) \, |f(s_k) - f(a)| < \varepsilon \qquad \blacksquare$$

<u>Theorem (Kirschbraun)</u>

Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ Lip then there exists $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$ s.t.

(i) $\bar{f} = f$ on $A$

(ii) $\mathrm{Lip}(f) = \mathrm{Lip}(\bar{f})$

<u>Proof:</u> 1932, Studia Math.

<u>Excersize:</u> Proof with $(\tilde{ii})$ $\mathrm{Lip}(\bar{f}) \leq \sqrt{m}\, \mathrm{Lip}(f)$

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**Theorem:** Let $\mu$ - Borel, regular measure on $\mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^m$ $\mu$-measurable, $A \subset \mathbb{R}^n$ $\mu$-m, $\mu(A) < \infty$

then

$\forall_{\varepsilon>0}$ $\underset{\substack{K\text{-compact}\\ K\subset A}}{\exists}$ s.t. (1) $\mu(A \setminus K) < \varepsilon$

(2) $f|_K$ - continuous

**Proof:**

$i \in \mathbb{N}$ and $\{B_{ij}\}_{j=1}^{\infty} \subset \mathbb{R}^n$, $\underset{j\neq k}{B_{ij} \cap B_{ik}} = \emptyset$, $\mathbb{R}^m = \bigcup_{j=1}^{\infty} B_{ij}$

$\operatorname{diam} B_{ij} < \frac{1}{i}$, then let $A_{ij} = A \cap f^{-1}(B_{ij})$

and $A = \bigcup_{j=1}^{\infty} A_{ij}$, $\nu = \mu|_A = \mu \llcorner A$

$\nu$ is Radon measure

in every $A_{ij}$ we can put $K_{ij}$ $K_{ij} \subset A_{ij}$, $K_{ij}$ - comp.

s.t. $\nu(A_{ij} \setminus K_{ij}) < \frac{\varepsilon}{2^{i+j}}$ then

$$\begin{aligned}
\mu\Big(A \setminus \bigcup_{j=1}^{\infty} K_{ij}\Big) &= \nu\Big(A \setminus \bigcup_{j=1}^{\infty} K_{ij}\Big) \\
&= \nu\Big(\bigcup_{j=1}^{\infty} A_{ij} \setminus K_{ij}\Big) \\
&= \sum_{j=1}^{\infty} \nu(A_{ij} \setminus K_{ij}) \\
&\leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^{i+j}} \\
&\leq \frac{\varepsilon}{2^i}
\end{aligned}$$

we can find $N(i)$ s.t.

$\mu\Big(A \setminus \bigcup_{j=1}^{\infty} K_{ij}\Big) < \frac{\varepsilon}{2^i}$ then define : $D_i = \bigcup_{j=1}^{N(i)} K_{ij}$

that $\forall_{i,j}$ $b_{ij} \in B_{ij}$ and $g_i : D_i \to \mathbb{R}^m$

$g_i(x) = b_{ij}$ for $x \in K_{ij}$ $(j \leq N(i))$

$|f(x) - g_i(x)| < \frac{1}{i} \quad \forall x \in D_i$

Let $K = \bigcap_{i=1}^{\infty} D_i$ - compact

$\mu(A \setminus K) \leq \sum_{i=1}^{\infty} \mu(A \setminus D_i) < \varepsilon$

on $K$ $g_i \rightrightarrows f$ so $f$ is continuous on $K$

($g_i$ where continuous)

<u>Remark:</u>

Let $\mu$ - Borel regular measure on $\mathbb{R}^n$ and

$f: \mathbb{R}^n \to \mathbb{R}^n$ $\mu$ - measurable, $\mu(A) < \infty$.

let $\varepsilon > 0$ then

$\exists \bar{f}: \mathbb{R}^n \to \mathbb{R}^n$ continuous s.t.

$\mu\{x \in A \mid \bar{f}(x) \neq f(x)\} < \varepsilon$

<u>Theorem</u> (Egoroff)

Let $\mu$ - measure on $\mathbb{R}^n$ and $f_k: \mathbb{R}^n \to \mathbb{R}^m$ $\mu$-m,

$k = 1, 2, \ldots$

$A \cap \mathbb{R}^n$ $\mu$-m, $\mu(A) < \infty$, $f_k \to g$ $\mu$ - a.e. on $A$

Then

$\forall \varepsilon > 0$ $\exists$ $B$ - $\mu$-m s.t. (1) $\mu(A \setminus B) < \varepsilon$

$B \subset A$

(2) $f_k \rightrightarrows g$ on $B$

<u>Def:</u> $f_i \xrightarrow[i \to \infty]{} g$ $\mu$ - a.e. if $f_k(x) \to g(x)$ except

$x \in C$, $\mu(C) = 0$.

<u>Def:</u> $f_k \to g$ in measure if

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mu(x \mid |f_n(x) - g(x)| > \varepsilon) = 0$$

<u>Proof</u> (of Egoroff):

Let $C_{ij} = \bigcup_{k=j}^{\infty} \{x \mid |f_k(x) - g(x)| > 2^{-i}\}$

$C_{i,j+1} \subset C_{i,j}$ , $\mu(A) < \infty$

$$\lim_{j \to \infty} \mu(A \cap C_{i,j}) = \mu\left(A \cap \bigcap_{j=1}^{\infty} C_{i,j}\right) = 0$$

$\exists_{N(i)}$ s.t. $\mu(A \cap C_{i,N(i)}) < \frac{\varepsilon}{2^i}$

let $B = A \setminus \bigcup_{i=1}^{\infty} C_{i,N(i)}$

then $\mu(A \setminus B) \leq \sum_{i=1}^{\infty} \mu(A \cap C_{i,N(i)}) < \varepsilon$

$\forall\ i \in \mathbb{N}$ $\forall_{x \in B}$ and $\forall\ n \geq N(i)$

$|f_n(x) - g(x)| \leq 2^{-i}$

$f_n \rightrightarrows g$ on $B$ ■

<u>Theorem</u>

1) If $f_n \to g$ $\mu$-a.e. then $f_n$ converge locally in $\mu$ ( locally means that it converges in every set of finite measure)

2) If $f_n$ converge locally in m. $\mu$ $\Rightarrow$ $\exists$ subsequence $n_k$ s.t. $f_{n_k} \to g$ $\mu$- a.e.

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Theorem: Let $0 < p < +\infty$, $f, f_n \in L^p(X, \mu, \mathbb{R})$ $\forall n \in \mathbb{N}$

Then the following conditions are equivalent

(a) $f_n \xrightarrow{L^p} f$

(b): (i) $f_n \to f$ in measure

(ii) $\forall \varepsilon > 0$ there exists $\mu$-m $E$ $\mu(E) < \infty$

s.t. $\sup\limits_{n\in\mathbb{N}} \left\{ \int\limits_{X\setminus E} |f_n|^p d\mu \right\} < \varepsilon$

(iii) $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\sup\limits_{\substack{A\ \mu\text{-m} \\ \mu(A) < \delta}} \left\{ \sup\limits_{n\in\mathbb{N}} \left\{ \int\limits_A |f_n|^p d\mu \right\} \right\} < \varepsilon$

UNIFORM INTEGRABILITY

(1) $\forall \varepsilon$ $\exists R_\varepsilon$ s.t.

$$\sup_{n\in\mathbb{N}} \left\{ \int\limits_{\{x : |f(x)|^p \geq R_\varepsilon\}} |f|^p d\mu \right\} < \varepsilon$$

HIGHER INTEGRABILITY CONDITION

(2) There exists nonnegative convex function $\Phi$ s.t.

$$\Phi : [0, +\infty) \to [0, +\infty) \qquad \lim_{x\to+\infty} \frac{\Phi(x)}{x} = +\infty$$

and $$\sup_{n\in\mathbb{N}} \int\limits_X \Phi(|f_n(x)|^p)\, d\mu < +\infty$$

Proof: $(1 \leq p < +\infty)$

$$\left| \left(\int\limits_B |f_n(x)|^p d\mu\right)^{1/p} - \left(\int\limits_B |f(x)|^p d\mu\right)^{1/p} \right| = \left| \|f_n \chi_B\|_p - \|f \chi_B\|_p \right| \leq \|(f_n - f)\chi_B\|_p \leq$$

$$\leq \|f - f_n\|_p \xrightarrow{n\to\infty} 0$$

(b) $\Rightarrow$ (a) $(1 \leq p < +\infty)$

$\forall \varepsilon > 0$ $\exists N_\varepsilon$ s.t. $k, n \geq N_\varepsilon$ then

$$\int\limits_X |f_k - f_n|^p d\mu < \varepsilon$$

$$\int\limits_X |f_n - f_k|^p d\mu \leq \underbrace{2^p \int\limits_{X\setminus E} (|f_n|^p + |f_k|^p) d\mu}_{(i)} + \underbrace{2^p \int\limits_{\substack{E\setminus B \\ \mu(E\setminus B) < \delta}} (|f_n|^p + |f_k|^p) d\mu}_{(iii)} + \underbrace{\int\limits_B |f_n - f_k|^p d\mu}_{(i)}$$

## SIGNED MEASURES

Def: $\nu : \mathcal{Z} \to \bar{\mathbb{R}} (= [-\infty, +\infty)$ or $(-\infty, +\infty])$ is $\boxed{\sigma\text{- finite}}$ signed measure if

(i) $\nu(\emptyset) = \emptyset$

(ii) if $A = \bigcup_{n=1}^{\infty} A_n$ disjoint $A_n \subset \mathcal{Z}$ then

$$\nu(A) = \sum_{n=1}^{\infty} \nu(A_n)$$

(iii) $Im(\nu(\mathcal{Z})) \subset (-\infty, +\infty]$ or $[-\infty, +\infty)$

$\boxed{iii}$ There exists $\{E_n\}_{n\in\mathbb{N}}$ s.t. $X = \bigcup_{n=1}^{\infty} E_n$ $|\nu(E_n)| < \infty$.

Remark:

(1) If $A \in \mathcal{Z}$ $|\nu(A)| < +\infty$ then $|\nu(B)| < +\infty$ $\forall B \subset \mathcal{Z}$ $B \subset A$

(2) If $A = \bigcup_{n=1}^{\infty} A_n$ $A_n \in \mathcal{Z}$ disjoint then $\sum |\nu(A_n)|$ is convex.

(3) $A_n \in \mathcal{Z}$ ~~$A_1 \subset \dots \subset A_{n-1} \subset A_n \subset \dots$~~ $\bigcup_{n=1}^{\infty} A_n = A$ then

$$\nu(A_n) \xrightarrow{n\to\infty} \nu(A)$$

(4) $A_n \in \mathcal{Z}$ $A_1 \supset A_2 \supset \dots$

$\bigcap_{n=1}^{\infty} A_n = B$ $|\nu(A_1)| < +\infty$ then $\nu(A_n) \xrightarrow{n\to\infty} \nu(B)$

### Hahn decomposition

Definition:

(1) Set $P \in \mathcal{Z}$ is $\nu$- positive if $\forall A \subset \mathcal{Z};\ A \subset P$ $\nu(A) \geq 0$

(2) Set $N \in \mathcal{Z}$ is $\nu$- negative if $\forall A \subset \mathcal{Z}$, $A \subset N$ $\nu(A) \leq 0$

(3) Set $Q \in \mathcal{Z}$ is $\nu$- zero if $\forall A \subset X$, $A \subset Q$, $\nu(A) = 0$

Theorem (Hahn 1921)

For signed measure there exists decomposition of $X$ s.t. $P \cap N = \emptyset$ $P \cup N = X$ and this decomposition is unique up to $Q$.

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Proof:

Step 1: claim : If $\nu : \mathcal{Z} \to [-\infty, +\infty)$ $\forall A \in \mathcal{Z}$ s.t. $\nu(A) > -\infty$ there exists set $P$ $\nu$- positive $P \subset A$ and $\nu(P) \geq \nu(A)$.

~~Proof~~ $\forall \varepsilon > 0$ $\exists A_\varepsilon \subset A$ s.t. $\nu(A_\varepsilon) \geq \nu(A)$ and $\nu(B) \geq -\varepsilon$ $\forall B \subset A_\varepsilon$.

$C \subset A$ $(C \in \mathcal{Z})$ there exist $B$ s.t. $\nu(B) \leq -\varepsilon$.

$\{B_k\}_{k\in\mathbb{N}}$ s.t. $B_k \cap B_i \neq \emptyset$ s.t. $B_k \subset A$, $\nu(B_k) \leq -\varepsilon$

$$\sum_{k\in\mathbb{N}} \nu(B_k) = \nu\Big(\bigcup_{k\in\mathbb{N}} B_k\Big)$$

$$\wedge\!\!|$$

$$\sum_{k\in\mathbb{N}} -\varepsilon = -\infty$$

$A_{\frac{1}{n}} \in Z$ and $P = \bigcap_{n \in \mathbb{N}} A_{\frac{1}{n}}$ $\nu$- positive

$\nu(P) \geq \nu(A)$

Step 2:

$Im(\nu) \subset [-\infty, +\infty)$

$\alpha = \sup \{ \nu(A) : A \in Z, \nu(A) \neq -\infty \}$

$\{P_n\}_{n \in \mathbb{N}}$ $\nu$- positive st. $\nu(P_n) \xrightarrow{n \to +\infty} \alpha$

$P = \bigcup_{n \in \mathbb{N}} P_n$ it is $\nu$ - positive and $\underbrace{\lim_{n \to \infty} \nu(P_n)}_{= \alpha}$ hmm

$\Rightarrow \nu(P) = \alpha$

Step 3:

$N = X \setminus P$, $N$ is $\nu$- negative

indeed:

$B \in N$, $\nu(B) > 0 \Rightarrow \nu(P \cup B) > \alpha$ $\lightning$

$\blacksquare$

Definition:

$\nu : Z \to \overline{\mathbb{R}}$, $X = P \cup N$ - Hahn decomposition. Then

$\nu^+ : Z \to [0, +\infty]$ is s.t.

$\nu^+(A) = \nu(A \cap P) = \nu L P$

similarly

$\nu^- : Z \to [0, +\infty]$ and $\nu^-(A) = -\nu(A \cap N) = -\nu L N$

Note

$\nu(A) = \nu^+(A) - \nu^-(A)$

Def: Total variation of the measure

$|\nu| : Z \to [0, +\infty]$, $|\nu|(A) = \nu^+(A) + \nu^-(A)$

Theorem:

1) $\nu^+(A) = \sup \{ \nu(B) : B \subset Z, B \subset A \}$

2) $\nu^-(A) = -\inf \{ \nu(B) : B \subset Z, B \subset A \}$

3) $|\nu|(A) = \sup \{ \sum_{j=1}^{n} |\nu(A_j)| : A_1, \dots, A_n$ - disjoint, $A_1 \cup \dots \cup A_n = A \}$

Def: We say that $\nu$ and $\mu$ are singular to each other if there exist decomposition $X = A \cup B$, $A \cap B = \emptyset$, $A, B \in Z$ s.t. $A$ is $\nu$- zero, $B$ is $\mu$- zero $(\nu \perp \mu)$

Theorem:

$\nu = \nu^+ - \nu^-$, $\nu^+ \perp \nu^-$

$\varsigma, \sigma$ $\nu = \varsigma - \sigma \Rightarrow \nu^+ \leq \varsigma$, $\nu^- \leq \sigma$.

Proof (only of the minimality of the decomposition):

$\nu^+(A) = (\nu LP)(A) = (\varrho LP)(A) - (\sigma LP)(A) \leq (\varrho LP)(A) \leq \varrho(A)$

The same for $\nu^-$

Def: $\|\nu\| = |\nu|(X)$

$\nu$ and $\mu$ Radon measures on $\mathbb{R}^n$

Def:

$$\overline{D}_\mu \nu(x) = \begin{cases} \limsup\limits_{r\to 0^+} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r)) > 0 \quad \forall r>0 \\ 0 & \text{if } \mu B(x,r) = 0 \quad \text{for } r>0. \end{cases}$$

$$\underline{D}_\mu \nu(x) = \begin{cases} \liminf\limits_{r\to 0^+} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r)) > 0 \quad \forall r>0 \\ 0 & \text{if } \mu(B(x,r)) = 0 \quad \text{for } r>0 \end{cases}$$

Def: $\nu$ is differentiable with respect to $\mu$ in $X$ if $\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) < +\infty$, then we wenote $\overline{D} = \underline{D} = D$.

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Def: we say that $\nu$ is differentiable w.r.t. $\mu$ in point $x$ if $\overline{D}_\mu\nu(x) = \underline{D}_\mu \nu(x) < +\infty$, then $\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) = D_\mu \nu(x)$ - derivative (density) of $\nu$ w.r.t. $\mu$.

Remarks: In the definition of upper and lower derivative it doesn't matter if the balls are open or closed.

Lemma 1: Fix $0 < \alpha < \infty$ then

(i) $A \subset \{x \in \mathbb{R}^n : \underline{D}_\mu \nu(x) \leq \alpha\} \Rightarrow \nu(A) \leq \alpha\mu(A)$

(ii) $A \subset \{x \in \mathbb{R}^n : \overline{D}_\mu \nu(x) \geq \alpha\} \Rightarrow \nu(A) \geq \alpha\mu(A)$

Remark: The set $A$ need not be $\mu$ or $\nu$ - measurable here.

Proof: WLOG we can assume that $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$. In other case we may restrict the measure to $\overline{B(0,R)}$.

**Lemma 2:** Let $\mu$ be a Borel measure on $\mathbb{R}^n$, $\mathcal{F}$ - the family of non-degenerate ($r \neq 0$) closed balls. Denote by $A$ a set of centers of balls from the family $\mathcal{F}$.
Assume that $\mu(A) < +\infty$ and $\forall a \in A$ $\inf \{ r \mid B(a,r) \in \mathcal{F} \} = 0$.
Then for each open set $\mathcal{U} \subset \mathbb{R}^n$ there exist a countable subfamily $G$ of disjoint balls from $\mathcal{F}$ s.t

1) $\bigcup_{B \in G} \subset \mathcal{U}$

2) $\mu((A \cap \mathcal{U}) \setminus \bigcup_{B \in G}) = 0$

Proof: Follows from Besicovitch covering theorem

Proof of lemma 1:
Fix $\varepsilon > 0$, Let $\mathcal{U}$ be an open set satisfying assumption of lemma 2

Define $\mathcal{F} = \{ B \mid B$ - closed balls centered in the points of $A$, $B \subset \mathcal{U}$, $\nu(B) \leq (\alpha + \varepsilon) \mu(B) \}$

Then $\inf \{ r \mid B(a,r) \in \mathcal{F} \} = 0$ $\forall a \in A$. So by lemma 2 there exists a countable, disjoint family of closed balls satisfying $\nu(A \setminus \bigcup_{B \in G} B) = 0$ then

$$\nu(A) \leq \sum_{B \in G} \nu(B) \leq (\alpha + \varepsilon) \sum_{B \in G} \mu(B) \leq (\alpha + \varepsilon) \mu(\mathcal{U})$$

but recall that then the measure of an arbitrary set $B$ is an inf of measures of open sets containing this set.

Thus $\nu(A) \leq (\alpha + \varepsilon) \mu(A)$

The proof of (ii) follows as (i).

**Theorem:** Let $\mu, \nu$ be radon measures on $\mathbb{R}^n$. Then $D_\mu \nu$ exists and is finite $\mu$ a.e. Moreover $D_\mu \nu(x)$ is $\mu$-measurable.

Proof: 1) $D_\mu \nu(x)$ exists and is finite $\mu$ a.e.

Let $I = \{ x \mid \bar{D}_\mu \nu(x) = +\infty \}$ Then $\forall \alpha > 0$
$I \subset \{ x \mid \bar{D}_\mu \nu(x) \geq \alpha \}$ and from previous lemma 1

$\mu(I) \leq \frac{1}{\alpha} \nu(I)$ passing with $\alpha \to \infty$

$\mu(I) = 0$ . Hence $\bar{D}_\mu \nu(x)$ is bounded $\mu$ a.e.

Define $R(a,b) = \{ x \mid \underline{D}_\mu \nu(x) < a < b < \bar{D}_\mu \nu(x) < +\infty \}$

Similary as above

$b\mu(R(a,b)) \leq \nu(R(a,b)) \leq a\mu(R(a,b))$ so

$\mu(R(a,b)) = 0$ (because $b > a$)

of course $\{ x \mid \underline{D}_\mu \nu(x) < \bar{D}_\mu \nu(x) < +\infty \} = \bigcup_{\substack{0<a<b<+\infty \\ a,b - \text{rational}}} R(a,b)$ and

consequently $D_\mu \nu$ exists and is finite $\mu$- a.e. ■

2) For $\forall x \in \mathbb{R}^n$ and $r > 0$ ( $B(y,r)$, $B(x,r)$ are open balls)

$\liminf_{y \to x} \mu(B(y,r)) \geq \mu(B(x,r))$ ← we want to show this

Define $y_k \to x$ and $f_k = \mathbb{1}_{\{B(y_k,r)\}}$ , $f = \mathbb{1}_{\{B(x,r)\}}$

Then on $B(x,r)$ $f_k$ converges pointwise to $f$ (also $\mu$-a.e.)

<u>Remark:</u> $\mu(B(y_k,r) \setminus B(x,r)) \geq 0$ and

$g = \mathbb{1}_{\{B(x, r + \sup_{n \in \mathbb{N}} |x - y_n|)\}}$ - summable majorant

with help of dominated convergence thm we complete the proof.

<u>Remarks:</u> For closed balls $\liminf(\cdot) \geq (\ )$.

3) Because of step 2 $\forall r > 0$ (fixed)

$x \mapsto \mu(B(x,r))$
$x \mapsto \nu(B(x,r))$ are semi- continuous , so they're Borel meas.

Moreover $f_r(x) = \begin{cases} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r)) \neq 0 \\ +\infty & \text{if } \mu(B(x,r)) = 0 \end{cases}$

Since $f_r$ is $\mu$- measurable and $\lim_{r \to \infty} f_r = D_\mu \nu$ $\mu$-a.e.

Then $D_\mu \nu$ is $\mu$- measurable.

<u>Def:</u> we say that $\nu$ is absolutely continuous w.r.t $\mu$.

(notation $\nu \ll \mu$) iff $\mu(A) = 0 \Rightarrow \nu(A) = 0$ $\forall A \subset \mathbb{R}^n$

**Theorem:** (Radon, Nikodym)

Let $\nu, \mu$ be Radon measures on $\mathbb{R}^n$ satisfying $\nu << \mu$

then $\nu(A) = \int_A D_\mu \nu \, d\mu$ for each set $A$ - $\mu$ - measurable.

**Remark:** The proof would be shorter if we didn't show an effective way for computing the derivative.

**Proof:** **Step 1:** Let $A$ - $\mu$ - measurable, then there exists $B$ - borel set s.t. $A \subset B$, $\mu(B \setminus A) = 0$

$\underset{\text{by def}}{\Rightarrow}$ $\nu(B \setminus A) = 0$, so $A$ is $\nu$ - measurable

**Step 2:** Define $Z = \{ x \in \mathbb{R}^n \mid D_\mu \nu(x) = 0 \}$

$I = \{ x \in \mathbb{R}^n \mid D_\mu \nu(x) = +\infty \}$ of course $\mu(I) = 0$

and since $\nu << \mu \Rightarrow \nu(I) = 0$.

Then from one of the previous lemma

$\nu(Z) \leq \alpha \mu(Z)$ $\forall$ $\alpha > 0$

so $\nu(Z) = 0$ (if only $\mu(Z) < +\infty$)

(In case $\mu(Z) = +\infty$ then consider $Z \cap K_i$, $K_i$ - compact set's and $K_i \subset K_{i+1}$ ?)

Finally $\nu(Z) = 0 = \int_Z D_\mu \nu \, d\mu$ and $\nu(I) = 0 = \int_I D_\mu \nu \, d\mu$.

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**Lemma:** $0 < \alpha < \infty$

$A \subset \{x \in \mathbb{R}^n \mid \underline{D}_\mu \nu(x) \leq \alpha\} \Rightarrow \nu(A) \leq \alpha \mu(A)$

**Step 3:**

Let $A$ - $\mu$-m and $1 < t < +\infty$. Then for $m \in \mathbb{Z}$ we define:

$A_m = A \cap \{x \in \mathbb{R}^n \mid t^m \leq D_\mu \nu(x) < t^{m+1}\}$

Then $A_m$ are $\nu$ and $\mu$-measurable $(A_m \cap A_l = \emptyset \;\; m \neq l)$

$A \setminus \bigcup_{m=-\infty}^{\infty} A_m \subset \{Z \cup I \cup \{x \mid \overline{D}_\mu \nu(x) \neq \underline{D}_\mu \nu(x)\}\}$

and

$\nu(A \setminus \bigcup_{m=\infty}^{-\infty} A_m) = 0$

$$\nu(A) = \sum_{m=-\infty}^{+\infty} \nu(A_m) \leq \sum_{m=-\infty}^{+\infty} t^{m+1} \mu(A_m) = t \sum_{m=-\infty}^{\infty} t^m \mu(A_m) \leq t \cdot \sum_{m=-\infty}^{\infty} \int_{A_m} D_\mu \nu \, d\mu =$$

$$= t \int_A D_\mu \nu \, d\mu = t \int_A D_\mu \nu \, d\mu$$

**Theorem:**

Let $\nu, \mu$ - Radon measure on $\mathbb{R}^n$

**Then**

1) $\nu = \nu_{ac} + \nu_s$, $\nu_{ac}, \nu_s$ - Radon m

$\nu_{ac} \ll \mu$ $\quad \nu_s \perp \mu$

2) $D_\mu \nu = D_\mu \nu_{ac}$ and $D_\mu \nu_s = 0$ $\mu$ a.e.

and

$\nu(A) = \int_A D_\mu \nu \, d\mu + \nu_s(A) \quad \forall A$ - Borel

**Proof:**

1) $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < +\infty$

2) $\mathcal{E} = \{A \subset \mathbb{R}^n \mid A$ - Borel $\mu(\mathbb{R}^n \setminus A) = 0$

$B_k \subset \mathcal{E}$ s.t.

$\nu(B_k) - \frac{1}{k} \leq \inf_{A \in \mathcal{E}} \nu(A) \qquad k \in \mathbb{N}$

If $B = \bigcap_{k=1}^{\infty} B_k$ then

$\mu(\mathbb{R}^n \setminus B) \leq \sum_{k=1}^{\infty} \mu(\mathbb{R}^n \setminus B_k) = 0$

$\nu(B) = \inf_{A \in \mathcal{E}} \nu(A)$

$\nu_{ac} = \nu \llcorner B$

$\nu_s = \nu \llcorner (\mathbb{R}^n \setminus B)$

3) $\nu \llcorner B \ll \mu$

$\mu(A) \Rightarrow (\nu \llcorner B)(A)$

$A \subset B$

$\mu(A) = 0 \Rightarrow \nu(A) = 0$

$\mu(A) = 0, \quad \nu(A) > 0$

$\nu(B \setminus A) < \nu(B) = \inf_{A \in \mathcal{E}} \nu(A)$

$B \setminus A \in \mathcal{E}$ $\lightning$

$\nu_{ac} \ll \mu$

4) $\nu_s \perp \mu$

$\mu(\mathbb{R}^n \setminus B) = 0$

$\nu(B) = (\nu \llcorner (\mathbb{R}^n \setminus B))(B) = 0$ and $\nu_s \perp \mu$

5) $D_\mu \nu_{ac} = D_\mu \nu$

$\alpha > 0$

$C_\alpha = \{x \in B \mid D_\mu \nu_s(x) \geq \alpha\}$

$\alpha \mu(C_\alpha) \leq \nu_s(C_\alpha) = 0$

$D_\mu \nu_s = 0$

<u>Theorem:</u>

Let $\mu$ - Radon measure on $\mathbb{R}^n$

$f \in L^1_{loc}(\mathbb{R}^n, \mu)$ then

$\lim_{r \to 0^+} ⨍_{B(x,r)} f \, d\mu = f(x)$, where $⨍_E f \, d\mu = \frac{1}{\mu(E)} \int_E f \, d\mu$

<u>Lemma:</u>

$\mu$ - Radon on $\mathbb{R}^n$

$1 \leq p < +\infty$, $f \in L^p_{loc}(\mathbb{R}^n, \mu)$

$\lim_{r \to 0^+} ⨍_{B(x,r)} |f(y) - f(x)|^p \, d\mu = 0$

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## Fubini Theorem

$\mu$ - measure on $X$ and $\nu$ - measure on $Y$

$\mu \times \nu : 2^{X\times Y} \to [0,+\infty]$

$(\mu\times\nu)(S) = \inf \{ \sum_{i=1}^{\infty} \mu(A_i)\cdot\nu(B_i)$ s.t. $A_i$ - $\mu$-m, $A_i \subset X$, $B_i$ - $\nu$-m, $B_i \subset Y$
$S \subset \bigcup_{i=1}^{\infty} (A_i \times B_i) \}$

**Thm (Fubini)**

Let $\mu$ measure on $X$ and $\nu$ measure on $Y$ then

(i) $\mu\times\nu$ is regular

(ii) if $A\subset X$ $A$ - $\mu$-measurable and $B \subset Y$ is $\nu$-measurable, then $A\times B$ is $(\mu\times\nu)$-measurable and $(\mu\times\nu)(A\times B) = \mu(A)\cdot\nu(B)$

(iii) Let $S\subset X\times Y$ $\sigma$-finite w.r.t. $\mu\times\nu$ then

$S_y = \{x \mid (x,y)\in S\}$ $\mu$-mes for a.e. $y$

$S_x = \{y \mid (x,y)\in S\}$ is $\nu$-mes for a.e. $x$, $\mu(S_y)$

$\nu(S_y)$ are integrable

$(\mu\times\nu)(S) = \int_Y \mu(S_y)\, d\nu(y) = \int_X \nu(S_x)\, d\mu(x)$

(iv) If $f$ is $(\mu\times\nu)$ - integrable and $\sigma$-finite then

$y \mapsto \int_X f(x,y)\, d\mu(x)$ is $\nu$-integrable

$x \mapsto \int_Y f(x,y)\, d\nu(y)$ is $\mu$-integrable

$$\int_{X\times Y} f(x,y)\, d(\mu\times\nu) = \int_Y \left[\int_X f(x,y)\, d\mu(x)\right] d\nu(y) = \int_X \left[\int_Y f(x,y)\, d\nu(y)\right] d\mu(x)$$

we skip the proof

**Theorem (Slicing Lemma)**

Let $\mu$ nonnegative finite Radon measure on $\mathbb{R}^{n+m}$. By $\sigma$ we denote canonical projection $\mu$ on $\mathbb{R}^n$

$(\sigma(E) = \mu(E\times\mathbb{R}^m)$ for every Borel set $E\subset\mathbb{R}^n)$

Then for $\sigma$-a.e. $x\in\mathbb{R}^n$ there exists probability Radon, measure s.t.

(i) $x \mapsto \int_{\mathbb{R}^m} f(x,y)\, d\nu_x(y)$ is $\sigma$-mes $\quad \forall f\in C_b^0(\mathbb{R}^{n+m})$

(ii) $\int_{\mathbb{R}^{n+m}} f(x,y)\, d\mu(x,y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} f(x,y)\, d\nu_x(y)\, d\sigma(x)$

Proof:

1. Let $\{f_k\}_{k=1}^{+\infty}$ dense in $C_0(\mathbb{R}^m)$

$$\gamma^k(E) = \int_{E\times\mathbb{R}^m} f_k(y)\,d\mu(x,y) \qquad (\text{note } f_k \geq 0 \Rightarrow \gamma^k \geq 0\ !)$$

$\forall E$ - borel, $E \subset \mathbb{R}^n$

$\gamma^k << \sigma \qquad \gamma^k(E) \leq \sigma(E)\,\|f^k\|_\infty$

$$\int_{E\times\mathbb{R}^m} f^k(y)\,d\mu(x,y) = \gamma^k(E) = \int_E D_\sigma \gamma^k(x)\,d\sigma(x)$$

$x \longmapsto D_\sigma \gamma^k(x)$

2. $f \in C_0(\mathbb{R}^m)$. we may take a subsequence s.t. $\|f_{k_j} - f\|_\infty \xrightarrow{k_j\to\infty} 0$

Let define $\Gamma_x(f) = \lim_{k_j\to+\infty} D_\sigma \gamma^{k_j}(x)$

$x \longmapsto \Gamma_x(f)$ is $\sigma$ measurable, bounded

$\Gamma_x(f)$ - linear and $|\Gamma_x(f)| \leq \|f\|_\infty$

From Riesz representation theorem there exists exactly one Radon bounded, measure $\nu$ s.t.

$\Gamma(f) = \int f\,d\nu \qquad \forall f \in C_0^\infty$

$$\Gamma_x(f) = \int_{\mathbb{R}^m} f(y)\,d\nu_x \qquad \forall f \in C_0(\mathbb{R}^m)$$

$$\int_{E\times\mathbb{R}^m} f(y)\,d\nu_x(y)\,d\sigma(x) = \int_{E\times\mathbb{R}^m} f(y)\,d\mu(x,y) \qquad \forall f\in C_0(\mathbb{R}^m),\ E\text{ - borel}$$

$$\int_{\mathbb{R}^n} g(x)\int_{\mathbb{R}^m} f(y)\,d\nu_x(y)\,d\sigma(x) = \int_{\mathbb{R}^n\times\mathbb{R}^m} g(x)\cdot f(y)\,d\mu(x,y)$$
, $g \in C_0(\mathbb{R}^n)$ and $f(y) \in C_0(\mathbb{R}^m)$

Lemma: Let $X_1, X_2$ compact then the function in form

$\sum_{i=1}^{N} f_1^i(x_1)\, f_2^i(x_2) \quad (f_1^i \in C(X_1),\ f_2^i \in C(X_2))$ are dense in $C(X_1\times X_2)$

So we can approximate

$$\|f(x,y) - \sum_{i=1}^{N} f^i(x)\cdot g^i(y)\|_\infty \xrightarrow{N\to\infty} 0$$

21. XII. 2011

Covering theorem (Vitali)

$\hat{B}$ - with radius $\times 5$

**Def:**

(i) Family $\mathcal{F}$ of closed balls in $\mathbb{R}^n$ covering of the set $A \subset \mathbb{R}^n$ if

$$A \subset \bigcup_{B \in \mathcal{F}} B$$

(ii) Fine cover of the set A is a cover and $\inf \{ \text{diam } B : x \in B, B \in \mathcal{F} \} = 0$ $\forall x \in A$.

**Theorem:** Let F family of nondegenerated closed balls in $\mathbb{R}^n$ s.t.

$\sup \{ \text{diam } B \mid B \in \mathcal{F} \} < +\infty$

then there exists countable sub family $\mathcal{G}$:

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in G} \hat{B}$$

**Proof:**

1) Let $D = \sup \{ \text{diam } B : B \in \mathcal{F} \}$ and $\mathcal{F}_j = \{ B \in \mathcal{F} : D/2^j < \text{diam } B < D/2^{j-1} \}$

for $j = 1, 2, \ldots$

$G_i \subset \mathcal{F}_i$

a) $G_1$ is maximal family disjoint from $F_1$

b) $(i \to i+1)$ $G_{i+1}$ is maximal disjoint with $G_1, \ldots, G_i$ and in $\mathcal{F}_{i+1}$

c) $\mathcal{G} = \bigcup_{j=1}^{\infty} G_j$

**Claim:**

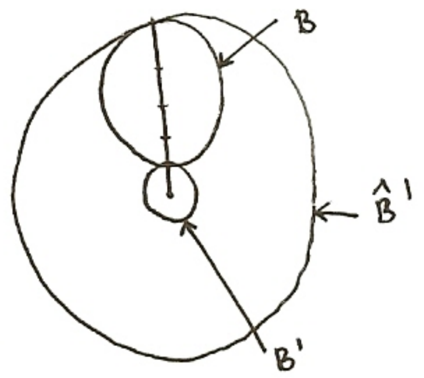
For every ball $B \subset \mathcal{F}$ there exists ball $B' \in \mathcal{G}$ s.t. $B \cap B' \neq \emptyset$ and $B \subset \hat{B'}$

**Proof:** (of claim)

There exists $j$ s.t. $B \in \mathcal{F}_j$ by maximality $G_j$ there exists ball $B' \subset \bigcup_{k=1}^{j} \mathcal{G}_k$ s.t. $B \cap B' \neq \emptyset$

$\Rightarrow$ $\text{diam } B' \geq D/2^j$ and $\text{diam}(B) \leq \frac{D}{2^{j-1}}$

$\text{diam } B \leq 2 \text{ diam } B'$



Claim:

If $\mathcal{F}$ is a fine cover of a set $A$ ( by closed ball)

and

$$\sup\{\operatorname{diam} B : B \in \mathcal{F}\} < +\infty$$

Then there exists countable subfamily $\mathcal{G}$ disjoint ball from $\mathcal{F}$ s.t. for every finite subset $\{B_1, \dots, B_m\} \subset \mathcal{F}$ we have :

$$A \setminus \bigcup_{k=1}^{m} B_k \subset \bigcup_{B \in \mathcal{G} \setminus \{B_1, \dots, B_m\}} \hat{B}$$

Proof:

Let $\mathcal{G}$ family from proof of previous theorem

Let $\{B_1, \dots, B_m\} \subset \mathcal{F}$ . If $A \subset \bigcup_{k=1}^{m} B_k$ - game over

Then let $x \in A \setminus \bigcup_{k=1}^{m} B_k$ but $\mathcal{F}$ is fine cover and there exist $B \in \mathcal{F}$ s.t. $x \in B$ and $B \cap B_k = \emptyset$ $(k = 1, \dots, m)$

but by previous consideration there exist $B' \in G$ s.t. $B \cap B' \neq \emptyset$ and $B \subset \hat{B}'$

■

Lemma: (about filling by disjoint balls)

Let $\mathcal{U} \subseteq \mathbb{R}^n$ open , $\delta > 0$ . Then there exists countable family of ~~balls~~ of disjoint closed balls in $\mathcal{U}$ s.t. $\operatorname{diam} B \leq \delta$ $\forall B \in \mathcal{G}$

and

$$\mathcal{L}^n(\mathcal{U} \setminus \bigcup_{B \in \mathcal{G}} B) = 0$$

Proof:

1) Let $1 - (\frac{1}{5})^{n} < \theta < 1$ moreover assume $\mathcal{L}^n(\mathcal{U}) < +\infty$

2) Claim there exists finite family $\{B_i\}_{i=1}^{n}$ disjoint $\subset \mathcal{U}$

s.t. $\operatorname{diam} B_i < \delta$ and

$$\mathcal{L}^n(\mathcal{U} \setminus \bigcup_{i=1}^{M_1} B_i) \leq \theta^1 \mathcal{L}^n(\mathcal{U})$$

Proof of the claim:

Let $\mathcal{F}_1 = \{B \mid B \subset \mathcal{U}, \operatorname{diam} B < \delta\}$ by previous consideration there exists countable family of disjoint balls $\mathcal{G}_1 \subset \mathcal{F}_1$ s.t.

$$\mathcal{U} \subset \bigcup_{B \in \mathcal{G}_1} \hat{B}$$

$$\mathcal{L}^n(\mathcal{U}) \leq \sum_{B \in \mathcal{G}_1} \mathcal{L}^n(\hat{B}) = 5^n \sum_{B \in \mathcal{G}_1} \mathcal{L}^n(B) = 5^n \mathcal{L}^n(\bigcup_{B \in \mathcal{G}_1} B)$$

$$\frac{1}{5^n} \mathcal{L}^n(\mathcal{U}) \leq \mathcal{L}^n(\bigcup_{B \in \mathcal{G}_1} B)$$

$$\mathcal{L}^n(\mathcal{U} \setminus \bigcup_{B \in \mathcal{G}_1} B) \leq \mathcal{L}^n(\mathcal{U}) - \frac{1}{5^n} \mathcal{L}^n(\mathcal{U})$$

3) $\mathcal{L}^n\left(U \setminus \bigcup_{i=1}^{M_k} B_i\right) \le \left(1-\left(\frac{1}{5}\right)^n\right)^k \mathcal{L}^n(U)$

## MAXIMAL FUNCTION

**Def:** Let $f \in L^p(\mathbb{R}^n)$ ~~denote~~ then

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\,dy$$

**Def:** $E_\alpha = \{x \in \mathbb{R}^n : (Mf)(x) > \alpha\}$

**Theorem:** If $f \in L^p(\mathbb{R}^n)$ then

(i) $\mathcal{L}^n(E_\alpha) \le \frac{A_1}{\alpha}\int_{\mathbb{R}^n} |f(x)|\,dx = \frac{A_1}{\alpha}\|f\|_{L^1}$ (for $p=1$)

(ii) $\|Mf(\cdot)\|_{L^p} \le A_p \|f\|_{L^p}$ for $(+\infty \ge p > 1)$

**Proof:**

$(Mf)(\cdot)$ is $\mathcal{L}^n$- mes

$$\sup_{\substack{r\in\mathbb{Q}\\ r>0}} \fint_{B(x,r)} |f(y)|\,dy$$

(i) $x \in E_\alpha$ Let $r_x > 0$ s.t. $\fint_{B(x,r_x)} |f(y)|\,dy > \alpha$

Clearly

$\bigcup_{x\in E_\alpha} \overline{B(x,r_x)}$ $\quad \sup_{x\in E_\alpha} r_x < +\infty$

Note $\fint_{B(x,r_x)} |f(y)|\,dy \le \frac{\|f\|_{L^1}}{\mathcal{L}^n(B(x,r_x))} \xrightarrow{r_x \to +\infty} 0$

we get countable family of the balls (closed and disjoint)

$\{B_k\}_{k\in\mathbb{N}}$ $\quad \sum_{k=1}^{\infty} \mathcal{L}^n(B_k) \ge \left(\frac{1}{5}\right)^n \mathcal{L}^n\left(\bigcup_{x\in E_\alpha} B(x,r_x)\right) \ge \left(\frac{1}{5}\right)^n \mathcal{L}^n(E_\alpha)$

$$\int_{B_k} |f|\,dy \ge \alpha\,\mathcal{L}^n(B_k)$$

$$\int_{\mathbb{R}^n} |f|\,dy \ge \int_{\bigcup_{n\in\mathbb{N}} B_n} |f(y)|\,dy = \sum_{k=1}^{+\infty}\int_{B_k} |f(y)|\,dy > \alpha\sum_{k=1}^{+\infty}\mathcal{L}^n(B_k) \ge \left(\frac{1}{5}\right)^n \cdot \alpha \cdot \mathcal{L}^n(E_\alpha)$$

This ends the case for $p=1$

11.01. 2012

Hausdorff measure

Def: Let $A \subset \mathbb{R}^n$, $0 \leq s < +\infty$, $0 < \delta < +\infty$ then

$$\mathcal{H}^s_\delta(A) = \inf\left\{\alpha(s)\sum_{j=1}^{+\infty}\left(\frac{\operatorname{diam} C_j}{2}\right)^s : A \subset \bigcup_{j=1}^{+\infty} C_j, \operatorname{diam} C_j < \delta\right\}$$

and $\alpha(s) = \dfrac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$ ; $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\,dx$, $0 < s < +\infty$

Def: $A \subset \mathbb{R}^n$, $s \in [0, +\infty)$

$$\mathcal{H}^s(A) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A)$$

Theorem: $\mathcal{H}^s$ is a borel, regular measure for $0 \leq s < +\infty$

Remark: It is not Radon because for $s < n$ $\underbrace{}_{\text{closed}}$ unit ball measure is $+\infty$

Proof:

1) $\mathcal{H}^s_\delta$ is a measure

Let $\{A_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and $A_k \subset \bigcup_{j=1}^{+\infty} C_j^k$

$\operatorname{diam} C_j^k \leq \delta$ then: $\{\{C_j^k\}_{j\in\mathbb{N}}\}_{k\in\mathbb{N}} \supset \bigcup_{k=1}^{+\infty} A_k$

therefore:

$$\mathcal{H}^s_\delta\left(\bigcup_{k=1}^{+\infty} A_k\right) \leq \sum_{k=1}^{+\infty}\sum_{j=1}^{+\infty} \alpha(s)\left(\frac{\operatorname{diam} C_j^k}{2}\right)^s$$

$$\mathcal{H}^s_\delta\left(\bigcup_{k=1}^{+\infty} A_k\right) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s_\delta(A_k)$$

$$\Rightarrow \mathcal{H}^s\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s(A_k)$$

2) $A, B \subset \mathbb{R}^n$ $\operatorname{dist}(A,B) > 0$, $\mathcal{H}^s(B) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(B)$

$0 < \delta < \frac{1}{4}\operatorname{dist}(A,B)$

$A \cup B \subset \bigcup_{k=1}^{+\infty} C_k$, $\operatorname{diam} C_k < \delta$

3) $\mathcal{H}^s$ borel regular measure

$\operatorname{diam} C = \operatorname{diam} \overline{C}$

Let $A \subset \mathbb{R}^n$ s.t. $\mathcal{H}^s(A) < +\infty \Rightarrow \mathcal{H}^s_\delta(A) < +\infty$

For $k \in \mathbb{N} \setminus \{0\}$ Let $\{C_j^k\}_{j=1}^{+\infty}$ s.t. $A \subset \bigcup_{j=1}^{\infty} C_j^k$

and $\sum_{j=1}^{\infty} \alpha(s) \left( \frac{\operatorname{diam} C_j^k}{2} \right)^s \leq \mathcal{H}^s_{\frac{1}{k}}(A) + \frac{1}{k}$

Let $A_k = \bigcup_{j=1}^{\infty} C_j^k$ $\qquad B = \bigcap_{k=1}^{\infty} A_k$

$A \subset A_k \implies A \subset B$

Theorem:

1) $\mathcal{H}^0$ · counting measure

2) $\mathcal{H}^1 = \mathcal{L}^1$ ~~and~~ on $\mathbb{R}^1$

3) $\mathcal{H}^s = 0$ on $\mathbb{R}^n$ for $s > n$

4) $\mathcal{H}^s(\lambda \cdot A) = \lambda^s \mathcal{H}^s(A)$ for $\lambda > 0$ and $A \subset \mathbb{R}^n$

5) $\mathcal{H}^s(L(A)) = \mathcal{H}^s(A)$, where L-

Proof:

1) $\alpha(0) = 1$ - obvious

2) obvious

3) $Q \subset \mathbb{R}^n$

$$\mathcal{H}^s_{\frac{\sqrt{n}}{m}}(Q) \leq \sum_{i=1}^{m^n} \alpha(s) \left( \frac{\sqrt{n}}{m} \right)^s = \alpha(s) n^{\frac{s}{2}} m^{n-s} \xrightarrow[m \to +\infty]{n-s<0} 0$$

Lemma:

Let $A \subset \mathbb{R}^n$ and $\mathcal{H}^s_\delta(A) = 0$ for some $\delta > 0$ then $\mathcal{H}^s(A) = 0$

Proof:

$s = 0$ $\quad \mathcal{H}^0_\delta(A) = 0 \implies A = \emptyset$

Let $s > 0$

$\mathcal{H}^s_\delta(A) = 0 \implies \forall \varepsilon > 0 \ \exists \{C_j\}_{j \in \mathbb{N}}$ s.t.

$A \subset \bigcup_{j=1}^{+\infty} C_j$ and $\sum_{j=1}^{\infty} \alpha(s) \left( \frac{\operatorname{diam} C_j}{2} \right)^s < \varepsilon$

$$\operatorname{diam} C_j \leq 2 \left( \frac{\varepsilon}{\alpha(s)} \right)^{1/s} = \delta(\varepsilon) \xrightarrow{\varepsilon \to 0^+} 0 \qquad \forall j \in \mathbb{N}$$

$\mathcal{H}^s_{\delta(\varepsilon)}(A) \leq \varepsilon$

$\mathcal{H}^s(A) \leq \varepsilon$

Lemma:

Let $A \subset \mathbb{R}^n$ and $0 < s < +\infty$ ($0 < s < t < +\infty$) then

(i) If $\mathcal{H}^s(A) < +\infty$ then $\mathcal{H}^t(A) = 0$

(ii) If $\mathcal{H}^t(A) > 0$ then $\mathcal{H}^s(A) = +\infty$

Proof:

Let $\mathcal{H}^s(A) < +\infty$ then fix $\delta > 0$

$\exists \{C_j\}_{j\in\mathbb{N}}$ s.t. $\operatorname{diam} C_j \le \delta$ $\quad A \subset \bigcup_{j=1}^{+\infty} C_j$

$$\sum_{j=1}^{+\infty} \alpha(s)\left(\frac{\operatorname{diam} C_j}{2}\right)^s \le \mathcal{H}^s_j(A) + 1$$
$$\le \mathcal{H}^s(A) + 1$$

$$\mathcal{H}^t_\delta(A) \le \sum_{j=1}^{+\infty} \alpha(t)\left(\frac{\operatorname{diam} C_j}{2}\right)^t = \frac{\alpha(t)}{\alpha(s)}\, 2^{s-t} \sum_{j=1}^{+\infty} \alpha(s)\left(\frac{\operatorname{diam} C_j}{2}\right)^s (\operatorname{diam} C_j)^{t-s}$$
$$\le \frac{\alpha(t)}{\alpha(s)}\, 2^{s-t}\, \delta^{t-s}\, (\mathcal{H}^s(A) + 1)$$

18.01.2012

$H^n(A) = \mathcal{L}^n(A)$ ; $A \subset \mathbb{R}^n$

Lemma: Let $f : \mathbb{R}^n \to [0, +\infty]$ $\mathcal{L}^n$- measurable. Then set

$A = \{ (x,y) \mid x \in \mathbb{R}^n,\ y \in \mathbb{R}$ s.t. $0 \le y \le f(x) \}$

is $\mathcal{L}^{n+1}$ measurable

Proof:

$g(x,y) = g(x) - y)$ $\quad \mathcal{L}^{n+1}$-meas $\quad$ ($y$ - $\mathcal{L}$ meas, $f(x)$ - $\mathcal{L}^n$)

and set $A' = \{(x,y) \mid y \ge 0\} \cap \{(x,y) \mid g(x,y) \ge 0\}$

$A'$ $\mathcal{L}^{n+1}$-m. $\quad A = A'$

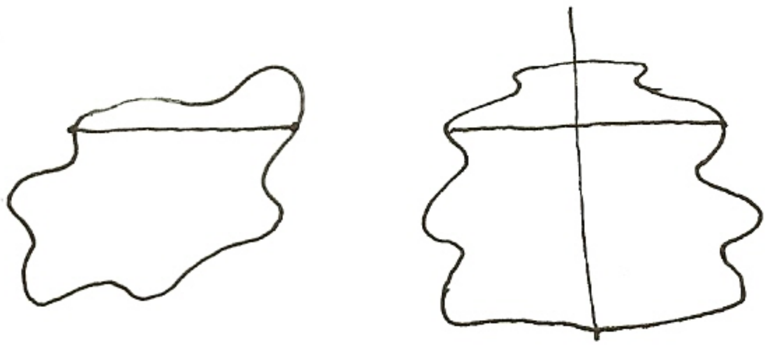
Steiner symmetrization:

Notation : $a, b \in \mathbb{R}^n$ , $|a| = 1$

Def: $L^a_b = \{ b + ta \mid t \in \mathbb{R} \}$

$P_a = \{ x \in \mathbb{R}^n \mid x \cdot a = 0 \}$

Idea:



we build symm. set s.t. measure of each line is preserved

Def: Let $a \in \mathbb{R}^n$, $|a| = 1$ and $A \subset \mathbb{R}^n$, $S_a = \bigcup_{b \in P_a} \{ b + ta \mid$

$|t| \le \frac{1}{2} \mathcal{H}^1(A \cap L^a_b)$ , $A \cap L^a_b \ne \emptyset \}$

**Lemma:**

(i) $\operatorname{diam} S_a(A) \le \operatorname{diam} A$

(ii) If $A$ is $\mathcal{L}^n$-m. then $S_a(A)$ is $\mathcal{L}^n$-m, moreover

$$\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$$

**Proof:** only for $a = e_n = (0, 0, \dots, 1)$ (general case holds because $\mathcal{H}$ measure is invariant to rotation)

$\mathcal{L}^1 = \mathcal{H}^1$ on $\mathbb{R}^1$

$f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ $\quad f(b) = \mathcal{H}^1(A \cap L^a_b)$ - use Fubini

$$\mathcal{L}^n(A) = \int_{\mathbb{R}^{n-1}} f(b)\, db$$ and then

$$S_a(A) = \{(b, y) \mid -\frac{f(b)}{2} \le y \le \frac{f(b)}{2}\} \setminus \{(b, 0) \mid L^a_b \cap A = \emptyset\}$$

$$\mathcal{L}^n(A) = \mathcal{L}^n(S_a(A))$$

**Theorem (isodiametric inequality)**

$$\mathcal{L}^n(A) \le \alpha(n) \left(\frac{\operatorname{diam} A}{2}\right)^n$$

n-dim ball

**Note:**

$\leftarrow$ diam A

**Proof:** Let assume $\operatorname{diam} A < \infty$ (why not :) )

Let $e_1, \dots, e_n$ - standard basis in $\mathbb{R}^n$

$A_i = S_{e_i}(A_{i-1})$ $\qquad A_0 = A$, $\quad A_n \overset{\text{def}}{=:} A^*$

**Claim:**

$x \in S^* \Rightarrow -x \in S^*$

**Proof of the claim:** obvious

**Claim:** $\mathcal{L}^n(A^*) \le \alpha(n) \left(\frac{\operatorname{diam} A^*}{2}\right)^n$

$$A^* \subset B\left(0, \frac{\operatorname{diam} A^*}{2}\right)$$

**Proof:**
$$\mathcal{L}^n(A) = \mathcal{L}^n(\bar{A}) = \mathcal{L}^n((\bar{A})^*) \le \alpha(n)\left(\frac{\operatorname{diam} (\bar{A})^*}{2}\right)^n$$
$$\le \alpha(n)\left(\frac{\operatorname{diam} \bar{A}}{2}\right)^n = \alpha(n)\left(\frac{\operatorname{diam} A}{2}\right)^n \qquad \square$$

Proof: ( $\mathcal{H}^n = \mathcal{L}^n$)

1) $\mathcal{L}^n(A) \leq \mathcal{H}^n(A) \quad \forall A \subset \mathbb{R}^n$

Let $\delta > 0$ and let $\{C_j\}_{j\in\mathbb{N}}$ s.t. $A \subset \bigcup_{j\in\mathbb{N}} C_j$ and $\operatorname{diam} C_j \leq \delta$

$$\mathcal{L}^n(A) \leq \sum_{j=1}^{\infty} \mathcal{L}^n(C_j) \leq \sum_{j=1}^{\infty} \alpha(n) \left( \frac{\operatorname{diam} C_j}{2} \right)^n$$

and $\mathcal{L}^n(A) \leq \mathcal{H}^n_\delta(A) \quad \forall \delta > 0$

$$\downarrow \delta \to 0$$

$$\mathcal{L}^n(A) \leq \mathcal{H}^n(A)$$

2) Without loss of generality we can restrict the covering to covering by cubes with size $< \delta$ in the definition of Lebesgue measure

3) $\mathcal{H}^n_\delta$ abs. cont. w.r.t. $\mathcal{L}^n$

Define $C_n = \alpha(n) \left(\frac{\sqrt{n}}{2}\right)^n$ and then

$$\alpha(n) \left( \frac{\operatorname{diam} Q}{2} \right)^n = C_n \mathcal{L}^n(Q)$$

$$\mathcal{H}^n_\delta(A) \leq \inf \left\{ \sum_{i=1}^{+\infty} \alpha(n) \left( \frac{\operatorname{diam} Q_i}{2} \right)^n \,\middle|\, A \subset \bigcup_{i=1}^{\infty} Q_i \ ; \ \operatorname{diam} Q < \delta \right\}$$

$$= C_n \mathcal{L}^n(A)$$

4) $\mathcal{H}^n(A) \leq \mathcal{L}^n(A)$ for $A \subset \mathbb{R}^n$

Let $\delta > 0$ and $\varepsilon > 0$

$A \subset \bigcup_{i=1}^{\infty} Q_i \qquad \operatorname{diam} Q_i < \delta$

Moreover

$$\sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) \leq \mathcal{L}^n(A) + \varepsilon$$

From Vitali covering theorem we know that there exists a family of disjoint balls: $\{B_k^i\}_{k\in\mathbb{N}}$

$B_k^i \subset \operatorname{int}(Q_i)$ s.t. $\operatorname{diam} B_k^i < \delta$

$\Rightarrow \mathcal{L}^n\left(Q_i \setminus \bigcup_{k=1}^{\infty} B_k^i\right) = \mathcal{L}^n\left(\operatorname{int} Q_i \setminus \bigcup_{k=1}^{\infty} B_k^i\right) = 0$ from Vitali

$\mathcal{H}^n$ is a.c. w.r.t. $\mathcal{L}^n$, so ($\mathcal{H}^n_\delta$ are $\leq \mathcal{H}^n$)

$\mathcal{H}^n_\delta \left( Q_i \setminus \bigcup_{k=1}^{\infty} B_k^i \right) = 0$ and

$$\mathcal{H}^n_\delta(A) \leq \sum_{i=1}^{\infty} \mathcal{H}^n_\delta(Q_i) = \sum_{i=1}^{\infty} \mathcal{H}^n_\delta \left( \bigcup_{k=1}^{\infty} B_k^i \right)$$

$$\leq \sum_{i=1}^{\infty} \sum_{k=1}^{+\infty} \mathcal{H}^n_\delta (B_k^i) \leq \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha(n) \left( \frac{\operatorname{diam} B_k^i}{2} \right)^n$$

balls are disjoint (we can write =)

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \mathcal{L}^n(B_k^i) = \sum_{i=1}^{\infty} \mathcal{L}^n \left( \bigcup_{k=1}^{\infty} B_k^i \right) = \sum_{i=1}^{\infty} \mathcal{L}^n(Q_i) \leq \mathcal{L}^n(A) + \varepsilon$$

$\varepsilon \to 0$, $\delta \to 0$ $\Rightarrow$ $\mathcal{H}^n = \mathcal{L}^n$ $\square$