30. XI. 2011

Lemma: $0 < \alpha < \infty$

$A \subset \{ x \in \mathbb{R}^n \mid \underline{D}_\mu \nu(x) \leq \alpha \} \Rightarrow \nu(A) \leq \alpha \mu(A)$

Step 3:

Let A - μ-m and $1 < t < +\infty$. Then for $m \in \mathbb{Z}$ we define:

$A_m = A \cap \{ x \in \mathbb{R}^n \mid t^m \leq D_\mu \nu(x) < t^{m+1} \}$

Then A_m are ν and μ-measurable $(A_m \cap A_l = \emptyset \quad m \neq l)$

$A \setminus \bigcup_{m=-\infty}^{\infty} A_m \subset \{ Z \cup I \cup \{ x \mid \overline{D}_\mu \nu(x) \neq \underline{D}_\mu \nu(x) \} \}$

and

$\nu (A \setminus \bigcup_{m=\infty}^{-\infty} A_m) = 0$

$$\nu(A) = \sum_{m=-\infty}^{+\infty} \nu(A_m) \leq \sum_{m=-\infty}^{+\infty} t^{m+1} \mu(A_m) = t \sum_{m=-\infty}^{\infty} t^m \mu(A_m) \leq t \cdot \sum_{m=-\infty}^{\infty} \int_{A_m} D_\mu \nu \, d\mu =$$

$$= t \int_A D_\mu \nu \, d\mu = t \int_A D_\mu \nu \, d\mu$$

Theorem:

Let ν, μ - Radon measure on $\mathbb{R}^n$

Then

1) $\nu = \nu_{ac} + \nu_s$, ν_{ac}, ν_s - Radon m

$\nu_{ac} << \mu$ $\qquad \nu_s \perp \mu$

2) $D_\mu \nu = D_\mu \nu_{ac}$ and $D_\mu \nu_s = 0$ μ a.e.

and

$\nu(A) = \int_A D_\mu \nu \, d\mu + \nu_s(A) \quad \forall A$ - Borel

Proof:

1) $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < +\infty$

2) $\mathcal{E} = \{ A \subset \mathbb{R}^n \mid A$ - Borel $\mu(\mathbb{R}^n \setminus A) = 0$

$B_k \subset \mathcal{E}$ s.t.

$\nu(B_k) - \frac{1}{k} \leq \inf_{A \in \mathcal{E}} \nu(A) \qquad k \in \mathbb{N}$

If $B = \bigcap_{k=1}^{\infty} B_k$ then

$\mu(\mathbb{R}^n \setminus B) \leq \sum_{k=1}^{\infty} \mu(\mathbb{R}^n \setminus B_k) = 0$

$\nu(B) = \inf_{A \in \mathcal{E}} \nu(A)$

$\nu_{ac} = \nu \llcorner B$

$\nu_s = \nu \llcorner (\mathbb{R}^n \setminus B)$

3) $\nu \llcorner B << \mu$

$\mu(A) \Rightarrow (\nu \llcorner B)(A)$

$A \subset B$

$\mu(A) = 0 \quad \Rightarrow \quad \nu(A) = 0$ $\not\Leftarrow$

$\mu(A) = 0, \quad \nu(A) > 0$

$\nu(B \setminus A) < \nu(B) = \inf_{A \in \mathcal{E}} \nu(A)$

$B \setminus A \in \mathcal{E}$ $\lightning$

$\nu_{ac} << \mu$

4) $\nu_s \perp \mu$

$\mu(\mathbb{R}^n \setminus B) = 0$

$\nu(B) = (\nu \llcorner (\mathbb{R}^n \setminus B))(B) = 0$ and $\nu_s \perp \mu$

5) $D_\mu \nu_{ac} = D_\mu \nu$

$\alpha > 0$

$C_\alpha = \{x \in B \mid D_\mu \nu_s(x) \geq \alpha\}$

$\alpha \mu(C_\alpha) \leq \nu_s(C_\alpha) = 0$

$D_\mu \nu_s = 0$

<u>Theorem:</u>

Let μ - Radon measure on $\mathbb{R}^n$

$f \in L^1_{loc}(\mathbb{R}^n, \mu)$ then

$\lim_{r \to 0^+} ⨍_{B(x,r)} f \, d\mu = f(x)$, where $⨍_E f \, d\mu = \frac{1}{\mu(E)} \int_E f \, d\mu$

<u>Lemma:</u>

μ - Radon on $\mathbb{R}^n$

$1 \leq p < +\infty$, $f \in L^p_{loc}(\mathbb{R}^n, \mu)$

$\lim_{r \to 0^+} ⨍_{B(x,r)} |f(y) - f(x)|^p \, d\mu = 0$