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Discrete dynamical systems

$f: X \to X$

$\{f^n\}_{n\in\mathbb{N}}$ - semigroup, if f is invertible then we have a group $\{f^n\}_{n\in\mathbb{N}}$

Examples:

*) X - topological (compact), f - continuous (homeo)

*) X - is C^r - manifold $(r \geq 1)$, f - C^r - diffeomorphism

*) X is measurable, f - measurable

Continuous dynamical systems

$\dot{x} = F(x)$, $x \in \mathbb{R}^n$, $F \in C^1$

$t \mapsto \varphi_t(x)$ solution of the equation such that $\varphi_0(x) = x$

$\{\varphi_t\}_{t\in\mathbb{R}}$ $\varphi_{t_1+t_2}(x) = \varphi_{t_1}(\varphi_{t_2}(x))$

Def: $x \in X$ $\{f^n(x)\}_{n\geq 0}$ trajectory (or orbit) of x. (forward)

$\{f^n(x)\}_{n\leq 0}$ trajectory of x (backward) if f^{-1} exists.

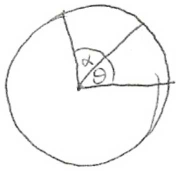
x periodic of period $p \in \mathbb{N}$ if $f^p(x) = x$.

basic period for the smallest p.

$f(x) = x \Rightarrow x$ is a fixed point

Example: $X = S^1 \subset \mathbb{C}$

$\simeq \mathbb{R}/\mathbb{Z}$



f rotation of angle α, $\alpha \in [0,1)$

$R_\alpha \cancel{f}(x) = z e^{2\pi i \alpha}$ complex notation, $z \in S^1$

$R_\alpha \cancel{f}(\theta) = \theta + \alpha$ (mod 1)

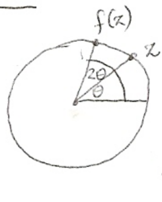
$x \in \mathbb{R}/\mathbb{Z}$

Question: what are periodic points of R_α:

1) $\alpha \in \mathbb{Q}$ then every x is periodic with period q $\alpha = \frac{p}{q}$

$\alpha \notin \mathbb{Q} \Rightarrow \forall n$ $R_\alpha^n(x) \neq x$

Example 2: $X = S^1$



$f(z) = z^2$

$f(\theta) = 2\theta$ (mod 1)

Periodic points of f:

$n \in \mathbb{N}$ $f^n(z) = z^{2n}$ $z^{2^n} = z$

$|z| = 1 \Rightarrow z \neq 0$

$z^{2^n - 1} - 1 = 0$

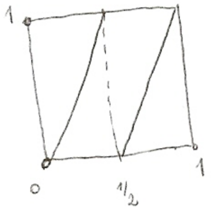
$\underline{z^{2^n} = 1}$

There are $2^n - 1$ periodic points of period n

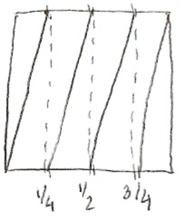
Corollary: The set of periodic points (of all periods) is dense in S^1

$\mathrm{Per}(f)$ - periodic points

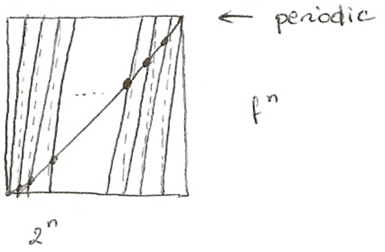
$\mathrm{Per}_p(f)$ - period points of period p.



$f(\theta) = 2\theta \pmod 1$



f^2



Def: Limit set $\omega(B)$ is the set of points $y \in X$ such that $x \in X$ there exists a sequence of $n_k \xrightarrow[k\to\infty]{} \infty$ with

$$f^{n_k}(x) \xrightarrow[k\to\infty]{} y$$

if f is invertible the we can define backward limit set $\alpha(x)$ analogously.

Q: what are the limit sets of point for rotation on the circle and $z \longmapsto z^2$ on the circle?

Rotation: if $\alpha \in \frac{p}{q}$ the $\omega(x) = \{R_\alpha^n\}_{n\geq 0} = \{x, R_\alpha(x), .., R_\alpha^{q-1}(x)\}$

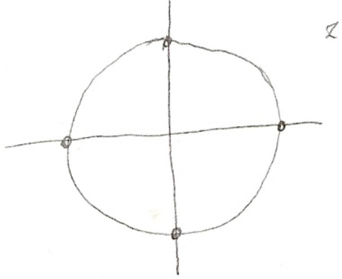
if $\alpha \notin \mathbb{Q}$ then $\forall x$ $\omega(x) = S^1$

Def: x is preperiodic if for some $n \geq 0$ $f^n(p)$ is periodic

$\{R_\alpha^n(x)\}_{n\geq 0}$ is infinite so $\exists y \in S^1$, $n_k \longrightarrow +\infty$ s.t.

$R_\alpha^{n_k}(x) \overset{\to}{\neq} y$

$\exists\ k, l$ s.t. $|R_\alpha^k(x) - R_\alpha^l(x)| < \varepsilon$
$k \geq l$



$z \longmapsto z^2$

1. $\mathrm{Per}(f)$ is dense
2. There exists a point with dense trajectory (i.e. $\omega(x) = S^1$)

$\theta \in [0, 1)$

$f(\theta) = 2\theta \mod 1$

$\Theta = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k}$ $\varepsilon_k \in \{0,1\}$ binary representation

$f(\Theta) = \sum_{k=1}^{\infty} \frac{2\varepsilon_k}{2^k} = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^{k-1}} = \sum_{k=0}^{\infty} \frac{\varepsilon_{k+1}}{2^k} = \varepsilon_1 + \sum_{k=1}^{\infty} \frac{\varepsilon_{k+1}}{2^k} \underset{\text{mod } 1}{=} \sum_{k=1}^{\infty} \frac{\varepsilon_{k+1}}{2^k}$

$$\begin{tikzcd} \Sigma_2^+ \arrow[r, "\sigma"] \arrow[d, "\pi"] & \Sigma_2^+ \arrow[d, "\pi"] \\ S^1 \arrow[r, "f"] & S^1 \end{tikzcd}$$

$\Theta \cong (\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots)$ shift map σ

$f(\Theta) \cong (\varepsilon_2, \varepsilon_3, \varepsilon_4, \dots)$

$\Sigma_2^+ = \{(\varepsilon_1, \varepsilon_2, \dots) : \varepsilon_k \in \{0,1\} \ \forall_k\}$

- symbolic space

$\pi \circ \sigma = f \circ \pi$

$(\underbrace{\varepsilon_1, \dots, \varepsilon_n}_{\text{fixed}}, \underbrace{*, \dots, *, \dots}_{\text{arbitrary}})$ cylinder

$\Theta_0 = (0\,1\ \ 00\,01\,10\,11\ \ 000\dots)$

↑ all finite sequences of length $n = 1, 2, 3, \dots$

Θ_0 has dense trajectory

Θ_0 has also dense set of preimages

Homework:

1. Show that a trajectory of a point x in a metric space X under a continuous map $f: X \to X$ is compact $\Leftarrow\Rightarrow$ x is periodic

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$f: X \to X$, $x \in X$ prove that $\{f^k(x)\}$ is compact $\Leftarrow\Rightarrow \exists N$ $f^N(x)$ is prepeniodic

X- metric,

$A = \{ s : \exists n_k \lim f^{n_k}(x) = f^s(x) \}$

m $\exists m_k$ $\lim f^{m_k}(x) = f^m(x)$

$\lim f^{m_k + l}(x) \overset{\text{continuity}}{=} f(\lim f^{m_k}(x)) = f^{m+l}(x)$

$\omega(f^m(x)) \supseteq \{f^n(x)\}_{n=m}$ (because $\{f^n(x)\}$ is closed)

$f^m(x), \dots, f^{m+l}(x), \dots$

suppose that $\omega(f^m(x))$ is finite (by contradiction), so we have limits

$f^{m+l}(x)$ $\lim f^{m_k + l}(x) = f^{m+l}(x)$

open set which contains $f^{m+l}(x)$ contains something more

$\text{int}\, f^{m+l}(x) = \emptyset$

$\overline{\{f^{m+l}(x)\}} = \{f^{m+l}(x)\}$

$\cup \{f^{m+l}(x)\}$ - complete but it's a x_0 sum of nowhere dense

2. 2^n, $n = 1, 2, 3, \dots$

$\underline{2}, \underline{4}, \underline{8}, \underline{1}6, \underline{3}2, \underline{6}4$

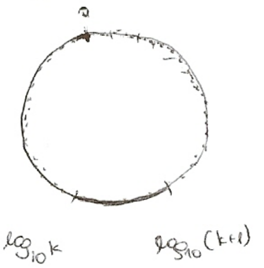
$1 \le 40$ 7 and 9 do not appear, do 7 and 9 ever accur?

if k is the first digit of 2^n, then $k \cdot 10^l \le 2^n < (k+1) \cdot 10^l$ for some l.

$\Rightarrow \log_{10} k + l \le n \cdot \log_{10} 2 < \log_{10}(k+1) + l$, $l \in \mathbb{N}$

(mod 1)

$n \cdot \log_{10} 2 \mod 1 \in [\log_{10} k, \log_{10}(k+1))$



$\{n\alpha\}$, $\alpha \notin \mathbb{Q}$ $\alpha = \log_{10} 2 \notin \mathbb{Q}$

from density for some n ~~these~~ $n\alpha \in [\log_{10} k, \log_{10}(k+1))$

3. <u>Definition</u>: $f: \mathbb{R} \to \mathbb{R}$, C^1, $f(x_0) = x_0$

- x_0 is attracting if $|f'(x_0)| < 1$
- x_0 is reapealing if $|f'(x_0)| > 1$
- x_0 is neutral if $|f'(x_0)| = 1$

4. Let $f: X \to X$, $g: Y \to Y$ continuous

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

if $\exists h$ - homeomorphism $h: X \to Y$

s.t. $h \circ f = g \circ f$

then we write $f \sim g$ (f, g are conjugate)

- observation 1: $\sim$ is an equivalence relation
- observation 2: $f \sim g$, then h maps orbits to orbits
- h preserve character of fix points

5. which of maps are conjugate. $f: \mathbb{R} \to \mathbb{R}$

$x = 0$

a) $f(x) = \frac{1}{2}x$ attracting

b) $f(x) = 2x$

c) $f(x) = 3x$

d) $f(x) = -2x$

b)–d) repelling

e) $f(x) = x^3$ ~~see~~ attracting, fix points $0, 1, -1$

let $h(x) = |x|^{\log_2 3} \cdot \operatorname{sgn} x$

$h(f(x)) = \operatorname{sgn} x \, |2x|^{\log_2 3} = 3 \operatorname{sgn} x \, |x|^{\log_2 3}$

$g(h(x)) = 3 \operatorname{sgn} x \, |x|^{\log_2 3}$

prove that $2x$ and $3x$ are not differentiably conjugate (no differentiable homeomorphism)

h^{-1} is not differentiable at $x=0$

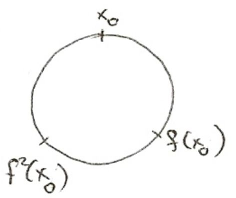
$h \circ f = g \circ h$, $h(0)=0$

$$\underset{h'(0)}{h'(f(0))} \cdot \underset{2}{f'(0)} = \underset{3}{g'(h(0))} \cdot h'(0) \quad \Rightarrow h'(0) \cdot 2 = 3 \cdot h'(0) \quad ↯$$

$(h^{-1})' = \infty$ $\quad h'(0) = 0$

what about d , it's not conjugete with any other

⇧



$\{\alpha + x\}$, $\alpha \notin \mathbb{Q}$

$[0,1)$

how many different arc lengths are there in $\{x_0, f(x_0), \dots, f^n(x_0)\}$?

6. Let $f: \mathbb{R} \to \mathbb{R}$ be continuous s.t. every $x \in \mathbb{R}$ is periodic_ prove that $f^2 = id$

f is „1-1" and stricly monotone

1° $x > y$ $\quad f(x) > f(y)$

$x \neq f(x)$

$x < f(x)$

$x = f^n(x) > x$ ↯

2° $x > y$ $\quad f(x) < f(y)$

$x < f(x)$

$f(x) > f^2(x)$

$f^3(x) \overset{\cancel{<}}{>} f^2(x)$

n - odd $\quad f^n(x) = x$

$f^{n+1}(x) < f^n(x) = x < f(x) = f^{n+1}(x)$ ↯

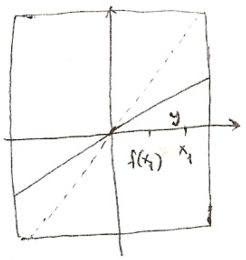
n - even

$f^2(x) = f^{n+2}(x) < f^{n+1}(x) = f(x)$

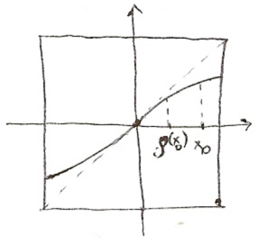
⇧ Find $f, g: [0,1] \to [0,1]$ such that $f^2 \sim g^2$, but $f \nsim g$

⇧ if $h: S^1 \to S^1$ is a homeomorphism of a circle which reverses orientation, then it has 2 fixed points

4.03.2011



$f(x) = \frac{x}{2}$



$g(x) = \sin x$

$I = [g(x_0), x_0]$

- odcinek fundamentalny

Budujemy sprzężenie h między f i g

Niech $h(x_0) = x_1$

Zadajemy h na I dowolnie, tak aby

$h(I) = J$

$h|I$ homeo zach. orientację

Definiujemy h_0 na $f(I)$

$x \in f(I)$

$h(x) = g h f^{-1}(x)$

Zauważmy, że $h(f(x)) = g h f^{-1}(f(x)) =$

$= g(h(x))$

Definiujemy h na $f^n(I)$ $n = 1, 2, 3, \ldots$

$h(x) = g^n h f^{-n}(x)$

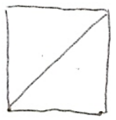
$x \in f^n(I)$

$h(f^n(I)) = g^n(J)$ homeo i zachodzi $hf = gh$

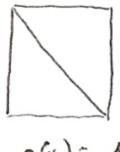
to samo robimy na lewej połosi.

<u>Zad</u> $f \sim g$ $f^2 \sim g^2$ (z pracy domowej)

I



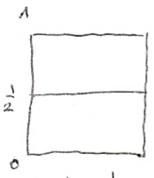
$f(x) = x$



$g(x) = 1-x$

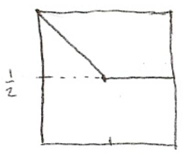
$g^2(x) = x$

II



$f(x) \equiv \frac{1}{2}$

$f^2(x) = \frac{1}{2}$

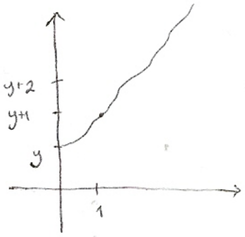


$g(x)$ $\frac{1}{2}$

$g^2(x) = \frac{1}{2}$

<u>Zad 2.</u>

$$\begin{tikzcd}
S^1 \arrow[r, "f"] & S^1 \\
\mathbb{R} \arrow[u, "\pi(x)=e^{2\pi i x}"] \arrow[r, "F"] & \mathbb{R} \arrow[u, "\pi"']
\end{tikzcd}$$



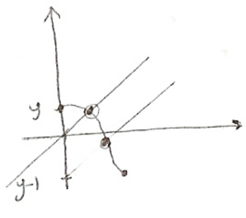
homeomorfizm S^1 zachowujący orientację

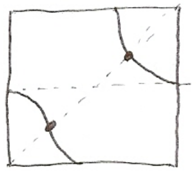
$F(x+1) = F(x) + 1$

$F(x+k) = F(x) + k$, $k \in \mathbb{Z}$

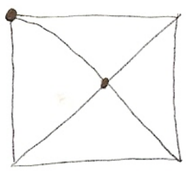
$F^n(x+k) = F^n(x) + k$, $n \in \mathbb{N}$

<u>Zad.</u> $f: S^1 \circlearrowleft$ u nas : homeomorfizm nie zachowuje orientacji





na rysunku widać, że ma dwa punkty (dokładnie) stałe.
Można by było się pokusić o argument



Tw (Ergodyczność obrotu na okręgu)

Jeśli $\alpha \notin \mathbb{Q}$ to trajektoria punktu $x \in S^1$ jest równorozłożona na S^1,

to znaczy

$\forall$ łuku $Ł$

$$\frac{\#\{k \in \{1,\dots,n\} : R_\alpha^k(x) \in Ł\}}{n} \longrightarrow \text{długość } Ł$$

(zakładamy, że długość okręgu jest 1)

Dowód: Dowodzimy, że dla każdej funkcji $\varphi : S^1 \to \mathbb{C}$, $\varphi \in L^1$ zachodzi

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(R_\alpha^k(x)) = \int \varphi \, dl \qquad (*)$$

$\uparrow$

unormowana miara Lebesgue'a na S^1

wystarczy pokazać (*) dla φ :
- f. charakterystycznych
- $\Downarrow$
- f. proste
- $\Downarrow$
- f. mierzalne nieujemne
- $\Downarrow$
- $f \in L^1$

} ogólny schemat

— || — dla φ ciągłych

— || — dla $\varphi(x) = c\, e^{2\pi i m x}$, $m \in \mathbb{Z}$,

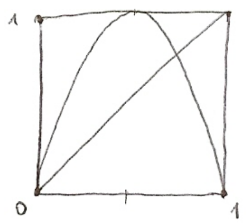
$m = 0$, to $\varphi(x) \equiv c$ lewa strona $\frac{1}{\cancel{n}} \cancel{n} \cdot c = c$ = prawa strona

$m \neq 0$

$$\left| \begin{matrix} \text{lewa strona} \\ x = 0 \end{matrix} \right| = \left| \frac{c}{n} \sum_{k=1}^{n} e^{2\pi i m k \alpha} \right| = \left| \frac{c}{n} e^{2\pi i m \alpha} \frac{1 - e^{2\pi i n m \alpha}}{\underset{\substack{\neq \\ 0 \text{ bo} \\ \alpha \text{- niewymierne}}}{1 - e^{2\pi i m \alpha}}} \right| \leq$$

$$\leq \frac{\text{const}}{n} 2$$

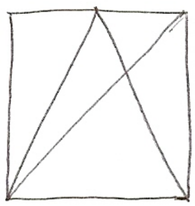
Przykład sprzężenia:



$f(x) = 4x(1-x)$

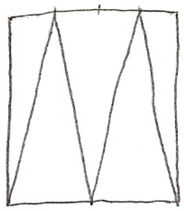
ogólnie przedstawiona rodzina parabol

$f_\lambda(x) = \lambda x(1-x)$ $\lambda \in (0,4]$



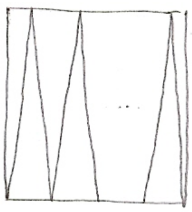
$g(x) = \begin{cases} 2x & x \leq \frac{1}{2} \\ 2-2x & x \geq 1/2 \end{cases}$

namiot „tent map"



g^2

…



2^n kawałków

g^n

⚠ Fakt: $f \sim g$ Wsk: skorzystać z tego, że $\sin^2 2\alpha = 4\sin^2\alpha\cos^2\alpha = 4\sin^2\alpha(1-\sin^2\alpha)$

11. 03. 2011

Fact: There exists a smooth (analytic) conjugation h between f and g i.e. an (analitic in (0,1)) homeo $h:[0,1] \to [0,1]$ s.t. $h \circ f = g \circ h$

$h(x) = \frac{2}{\pi} \arcsin \sqrt{x}$, $h^{-1}(x) = \sin^2(\frac{\pi}{2}x)$

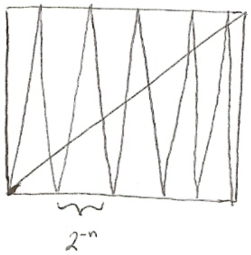
Proof: $h(f(h^{-1}(x))) = h(f(\sin^2(\frac{\pi}{2}x))) = h(4\sin^2(\frac{\pi}{2}x)(1-\sin^2(\frac{\pi}{2}x))) = h(4\sin^2(\frac{\pi}{2}x)\cos^2(\frac{\pi}{2}x)) =$

$= h(\sin^2(\pi x)) = \frac{2}{\pi}\arcsin(\sin \pi x) = \begin{cases} 2x & 0 \le x < \frac{1}{2} \\ 2-2x & \frac{1}{2} \le x \le 1 \end{cases}$

Cor: f has dense set of periodic points

Proof: It is enough to cheack that g has a dense set of periodic points.

g^n



g^n has two fixed points on each binary interval of the form $[k2^{-n}, (k+1)2^{-n}]$

so the assertion is obvious

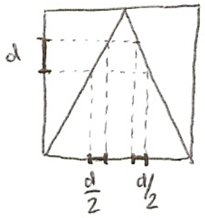
Def: A measure (finite, borel) μ is invariant with respect to f (f-invariant) if $\mu(f^{-1}(A)) = \mu(A)$ $\forall$ μ-measurable $A \subset X$

$f: X \to X$

$(X, \mathfrak{M}, \mu)$ measurable space

f μ-measurable in the sense $A \in \mathfrak{M} \Rightarrow f^{-1}(A) \in \mathfrak{M}$

Fact 1: (a) The Lebesgue measure is invariant for the tent map



(b) $z \mapsto z^2$, $z \in S^1$ has invariant Lebesgue measure

Fact 2: if h conjugates $f: X \to X$ and $g: Y \to Y$, μ is f-invariant, then $h_*\mu$ is g-invariant

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

$h_{*}\mu(A) := \mu(h^{-1}(A))$

Proof:
$$\begin{array}{ccc} h_{*\mu}(g^{-}(A)) & \overset{?}{=} & h_{*\mu}(A) \\ \shortparallel & & \| \\ \mu(h^{-1}g^{-1}(A)) & & \\ \shortparallel & & \\ \mu(f^{-1}h^{-1}(A)) & = & \mu(h^{-1}(A)) \end{array}$$

Cor:

h_*^{-1} (Leb) is invariant for the full parabole $x \mapsto 4x(1-x)$

$$\begin{tikzcd} \boxed{A} \arrow[r, bend left, "h"] & \boxed{h(A)} \arrow[l, bend left, "h^{-1}"] \end{tikzcd}$$

$$h_*^{-1}(\text{leb})(A) = \text{leb}(h(A)) = \int_{h(A)} 1 \, d\text{leb} = \int_A h'(x)dx$$

$$h(x) = \frac{2}{\pi} \arcsin\sqrt{x}$$

$$h'(x) = \frac{2}{\pi} \frac{1}{2\sqrt{1-x}\sqrt{x}} = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}$$

h_*^{-1} (leb) - the abs. continue measure with density $\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}$

Rotation number

$f: S^1 \to S^1$ homeo, preserving orientation

$F: \mathbb{R} \to \mathbb{R}$ lift of f

Def: Rotation number $\rho = \lim_{n\to\infty} \frac{F^n(x)}{n}$ mod 1

then ρ exists does not depend on F and x.

1. $\rho \in \mathbb{Q}$ <=> f has a periodic point

1'. $\rho = 0$ <=> f has a fixed point

2. $\rho(f^k) = k\rho(f)$ mod 1

3. If f is conjugate to g, then $\rho(f) = \rho(g)$

Proof: $f \sim g \Rightarrow F \sim G \Rightarrow G = HFH^{-1}$

$$\lim \frac{G^n(x)}{n} = \lim \frac{H(F^n(H^{-1}(x)))}{n} = \lim \underbrace{\frac{HF^n(H^{-1}x) - F^n(H^{-1}x)}{n}}_{0} + \underbrace{\frac{F^n(H^{-1}x)}{n}}_{\to \rho(f)}$$

because $H - Id$ is periodic
$(H - Id)(x+1) = (H - id)(x)$

so bounded

4. $f \longmapsto \rho(f)$ is continuous in the space of all orient. pres. homeo of S^1.

18.03.2011

Fact: $\mathcal{H}$ - the space of all orientation preserving homeo of S^1 with C^0 topology (uniform convergence topology) then $f \mapsto \rho(f)$ is continuous.

Proof: $f \in \mathcal{H}$, let $\varepsilon > 0$

Then $\exists \ \frac{p_+}{q_+}, \frac{p_-}{q_-} \in \mathbb{Q}$ s.t. $\rho(f) - \varepsilon < \frac{p_-}{q_-} < \rho(f) < \frac{p_+}{q_+} < \rho(f) + \varepsilon$

F - the lift of f $\quad F: \mathbb{R} \to \mathbb{R}$, $F(0) \in [0, 1)$

$\forall_n \ F^n(0) \in [0, n)$

$\frac{F^n(0)}{n} \in [0, 1)$

we take k s.t. $\frac{p_-}{q_-} < \frac{F^k(0)}{k} < \frac{p_+}{q_+}$

We can assume that q_-, q_+ devide k

Then $\frac{kp_+}{q_+} = m_+ \in \mathbb{N}$, $\frac{kp_-}{q_-} = m_- \in \mathbb{N}$

$\Rightarrow$ $m_- < F^k(0) < m_+$

we take g close to f (then G close to F)

$m_- < G^k(0) < m_+$

$$G^{2k}(0) = G^k(G^k(0)) \in (G^k(m_-), G^k(m_+)) = (G^k(0) + m_-, G^k(0) + m_+) \subset (2m_-, 2m_+)$$

By induction

$j \in \mathbb{N}$ $G^{jk}(0) \in (jm_-, jm_+)$

$$\boxed{\begin{aligned} \frac{m_+}{k} &< \rho(f) + \varepsilon \\ \frac{m_-}{k} &> \rho(f) - \varepsilon \end{aligned}}$$

~~Let~~

Let $n \in \mathbb{N}$ $n = jk + r$, $j \in \mathbb{N}$, $r \in \{0, \dots, k-1\}$

$$\frac{G^n(0)}{n} = \frac{G^r(G^{jk}(0))}{jk+r}$$ so

$$\frac{G^n(0)}{n} < \frac{G^r(jm_+)}{jk+r} = \frac{G^r(0) + jm_+}{jk+r} < \frac{G^r(0)}{jk+r} + \frac{jk(\rho(f)+\varepsilon)}{jk+r}$$

$\Rightarrow$ $\rho(f) < \rho(f) + \varepsilon$ the same estimates on the other side ■

$f: M \to M$ M - compact C^r, $r \geq 1$

$\{f^n\}_{n \in \mathbb{Z}}$ f diff C^r

$V: M \to TM$

C^r vector field

$\{\varphi_t\}$ flow

$f: I \to I$

f - continuous map , $I = [0,1]$

1. what periods (of periodic points) can appear for such f?

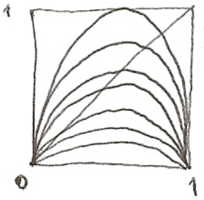
<u>Theorem</u> (Sharkowski)

we define an order in $\mathbb{N}$ (Sharkovski order) $\prec$

$3 \succ 5 \succ 7 \succ \dots \succ 2\cdot 3 \succ 2\cdot 6 \succ 2\cdot 7 \succ \dots \succ 2^2\cdot 3 \succ 2^2\cdot 5 \succ 2^2\cdot 7 \succ \dots \succ \dots \succ 2^n\cdot 3 \succ 2^n\cdot 5 \succ 2^n\cdot 7 \succ \dots$

$\dots \succ 2 \dots \succ 2^2 \succ 2^1 \succ 1$

2. If we have a continuous family $f_\lambda: I \to I$ of maps , in what way the possible periods appear if we change λ?



$f_\lambda(x) = \lambda x\ (1-x)$

$\lambda \in (0,4)$

f_4 conjugate to the tent map

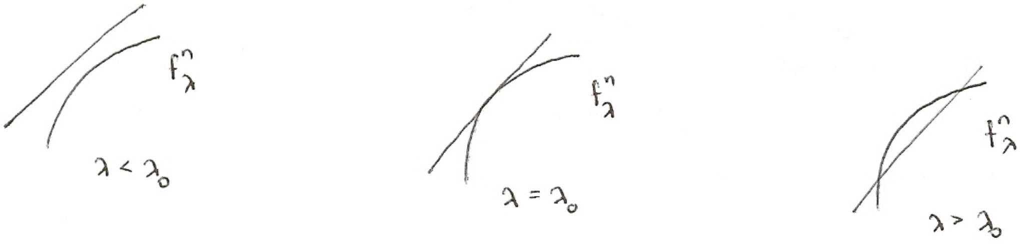
$\varepsilon \approx 0$ $\quad f_\varepsilon$ has only $\{1\}$ as a set of periods

f_4 has all periods

How new periods appear ?

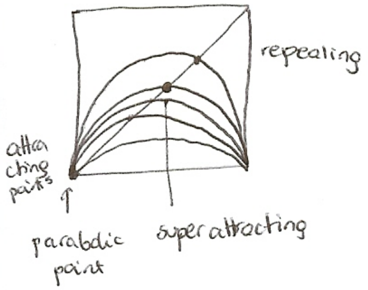
suppose that for λ_0 there appear period n

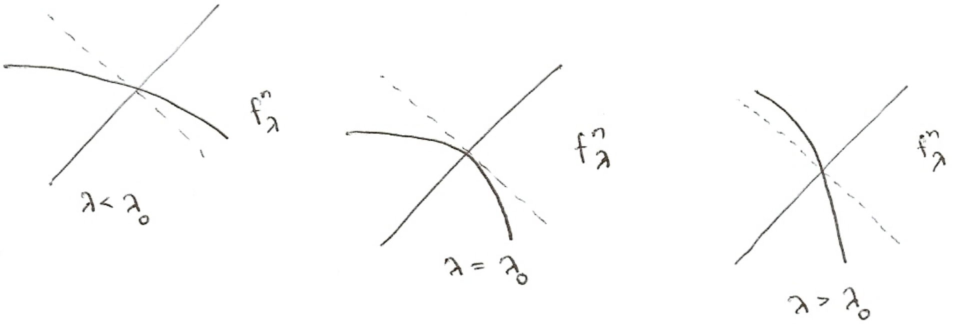
$f^n_{\lambda_0}(x_0) = x_0$



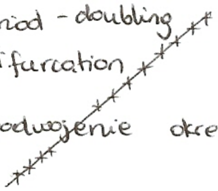
$\lambda < \lambda_0$ $\quad \lambda = \lambda_0$ $\quad \lambda > \lambda_0$

saddle-node bifurcation (siodło - węzeł bifurkacja)



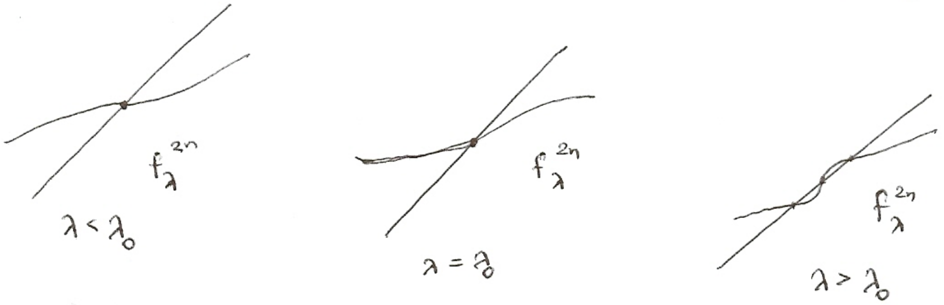


$\lambda < \lambda_0$ $\quad \lambda = \lambda_0$ $\quad \lambda > \lambda_0$

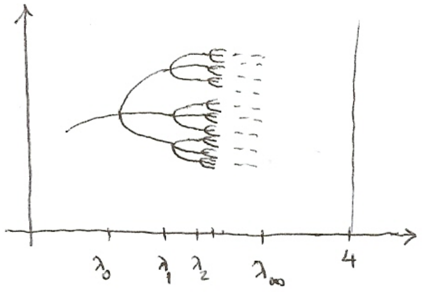


period-doubling bifurcation

(podwojenie okresu)



$\lambda < \lambda_0$ $\quad \lambda = \lambda_0$ $\quad \lambda > \lambda_0$



λ_n creation of period 2^{n+1}

Feigenbaum

λ_∞ Feigenbaum parameter

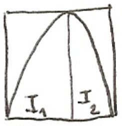
Theorem:

$\lambda_\infty - \lambda_n \sim q^n$ (up to a const)

$0 < q < 1$

q - Feigenbaum's constant

Thm: The constant q is universal for many families of unimodal maps



f unimodal if

$f \nearrow$ on I_1 $\quad$ smooth

$f \searrow$ on I_2

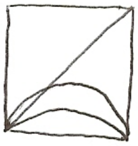
Renormalization is replacing f on some small interval by f^k

$f \longmapsto \mathcal{R}(f)$

Q : $\mathcal{R}^n(f) \xrightarrow{?} g$

Concider the quadratic family $f_\lambda(x) = \lambda x(\cdot 1 - x)$

Def: we say that f is hyperbolic, if there exists an attracting periodic orbit (ie $\exists x$, $f^n(x) = x$, $|(f^n)'(x)| < 1$)



Problem: Are hyperbolic maps dense in the quadratic family ?
G. Świątek & J. Graczyk, M. Lyubich
(The answer is yes)

25.03.2011

www.mimuw.edu.pl/~baranski - zadania z układów dynamicznych, po polsku

A part of Sharkovski theorem (Burns & Hasselblatt)

If a continuous map of the interval $I = [0,1]$ has a point of (minimal) period 3, then it has points of (minimal) periods $n = 1, 2, 3, 4, \dots$

Def: If J, K are intervals in I then we say that J f-covers K if $f(J) \supset K$. We write $J \to K$.

Lemma: If $J_0 \to J_1 \to J_2 \to \dots \to J_{n-1} \to J_0$ (*) then the is interval $K \subset J_0$ such that $f^n(K) = J_0$ and there is a point $x \in K$ such that $f^n(x) = x$ and $f^i(x) \in J_i$, $i = 0, \dots, n-1$ (point x follows the loop (*))

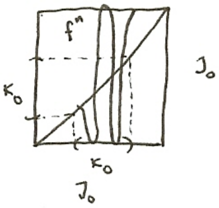
Proof: $J_{n-1} \to J_0$ so there exist a $K_{n-1} \subset J_{n-1}$ s.t. $f(K_{n-1}) = J_0$

$K_{n-1} \to J_0$, there is a $K_{n-2} \subset J_{n-2}$ st. $f(K_{n-2}) = K_{n-1}$

(because $J_{n-2} \to K_{n-1}$)

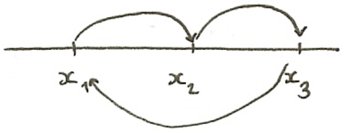
⋮

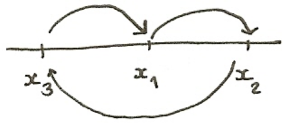
$K_0 \subset J_0$ s.t. $f(K_0) = K_1$. By construction $f^n(K_0) = J_0$, $K := K_0$

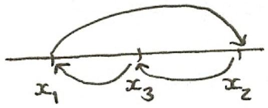


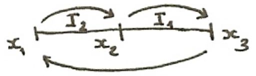
there is an interval $[a,b] \subset K_0$ s.t. $f^n(a), f^n(b)$ are both endpoints of J_0, so graph $(f^n|_{[a,b]})$ crosses the diagonal at $(x, f^n(x))$ we have $f^n x = x$

Suppose we have a point x_1 of minimal period 3







Assume 

(the other cases are symetric)

$I_1 \to I_1$

$I_1 \to I_2$ $\qquad \circlearrowleft I_1 \rightleftarrows I_2$

$I_2 \to I_1$

we find a point of period n, $n = 1, 2, 4, 5, \dots$

$\underline{n=1}$ $\quad I_1 \to I_2$ so by anything there is a point x s.t. $f(x) = x$.

$\underline{n=2}$ $\quad I_1 \to I_2 \to I_1$ by lemma there exist a point $x \in I_1$ s.t. $f^2(x) = x$.

If $f(x) = x$ the $x \in I_1 \cap I_2$, so $x = x_2$ but $f(x_2) \neq x_2$, so x has minimal period 2.

$\underline{n \geq 4}$ $\quad \underbrace{I_2 \to I_1 \to I_1 \to \dots \to I_1 \to I_2}_{n+1 \text{ length}}$ by lemma $\exists\, x \in I_2$ s.t. $f^n(x) = x$ it is obvious that n is minimal period

(if $f^k(x) = x$, $k < n$ then $x \in I_1 \cap I_2$

$\Rightarrow$ $x = x_2$

$x_2 \in I_1 \cap I_2$

$f(x_2) \in I_1 \setminus I_2$

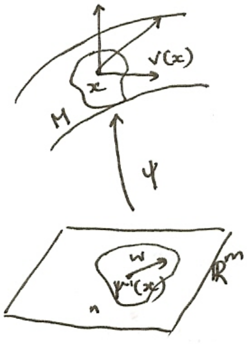
$f^2(x_2) \in I_2 \setminus I_1$ - $\lightning$ $(f^2(x) \in I_1)$)

M - C^r-manifold, $r \geq 1$, $\dim M = m$, $M \subseteq \mathbb{R}^n$

M - compact, usually connected

(I) $f: M \to M$ C^r-diff, $\{f^n\}_{n \in \mathbb{Z}}$

(II) $V: M \to TM$ C^r - vector field



If V is not tangent to M we take a tangent

$w(u) = (D\psi(\psi^{-1}(x)))^{-1}(V(x))$

$u = \psi^{-1}(x)$

$x = \psi(u)$

w C^r vector field on $\underset{\text{open}}{U \subset \mathbb{R}^m}$ $\qquad u' = w(u)$

$g_t(u)$ solution of $\dot{u} = w(u)$ s.t. $g_0(u) = u$ $\quad \{g_t\}_t$ flow $\quad g_{t_1} \circ g_{t_2} = g_{t_1 + t_2}$

$\Leftrightarrow$ $\frac{d}{dt} g_t(u) = w(g_t(u))$

$\varphi_t(x) = \psi(g_t(\psi^{-1}(x)))$

$\{\varphi_t\}$ is the flow for V $\quad t\in\mathbb{R}$

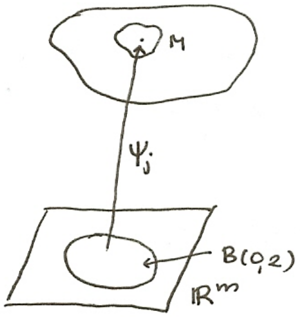
$\varphi_{t_1+t_2} = \varphi_{t_1}\circ\varphi_{t_2}$

$\frac{d}{dt}\varphi_t(x) = V(\varphi_t(x))$

Def: Topology in the space of C^r maps

$f\in C^r(M,\mathbb{R}^s)$

$$\|f\|_r = \max_{i=0,..,r}\ \max_{j=1,..,J}\ \max_{u\in B(0,1)} \|D^i f(\psi_j^{-1}(u))\|$$



$\{\psi_j\}$ an atlas of C^r parametrizations of M

$\psi_j : B(0,2) \longrightarrow M$

$\bigcup_j \psi_j(B(0,1)) = M$

Lemma 1: ~~$\mathrm{Diff}^r(M)$ (C^r diffeomorphisms of M)~~

Let U be open $U\subset\mathbb{R}^m$, $f: U\to\mathbb{R}^s$, $f\in C^r$, $K\subset U$ compact. Then

$\forall\varepsilon>0$ $\exists g:\mathbb{R}^m\to\mathbb{R}^s$ $g\in C^\infty$ s.t.

$\|(f-g)\|_{K,r} < \varepsilon$

take $\varphi:\mathbb{R}^m\to\mathbb{R}$, $\varphi\in C^\infty$ s.t. $\varphi\equiv1$ on K, $\varphi\equiv0$ outside a ~~ball~~

small neighbourhood of K.

Let $h=\varphi\cdot f$, $h\equiv f$ on K, $h\equiv0$ outside a small ngh of K.

$$\exists\delta>0 \quad \max_{i=0,\dots,r}\ \sup_{\substack{u\in K,\ v\in\mathbb{R}^m\\ \|v\|<\delta}} \|D^i h(u+v) - D^i h(u)\| < \varepsilon \quad (\star)$$

$\varphi_\delta:\mathbb{R}^m\to\mathbb{R}$ $\quad \varphi_\delta\in C^\infty$

$\varphi_\delta(v)\equiv0$ for $\|v\|>\delta$ $\quad \varphi_\delta\geq0$

$\int\varphi_\delta = 1$

Define g as the convolution

$$g(u) = \int_{\mathbb{R}^m}\varphi_\delta(v)h(u+v)dv = \underbrace{\int_{\mathbb{R}^m}\varphi_\delta(z-u)h(z)dz}_{C^\infty \text{ w.r. to } u}$$

$g\in C^\infty$ by general theorem on the derivation of integrals with parameter

$$\|D^i g(u) - D^i f(u)\|_{u\in K} = \|D^i g(u) - D^i h(u)\| = \left\|\int\varphi_\delta(v)D^i h(u+v)dv - \int\varphi_\delta(v)D^i h(v)dv\right\| \overset{(\star)}{\leq}$$

$$\leq \int\varphi_\delta(v)\cdot\varepsilon\, dv = \varepsilon$$

1. 04. 2011

M - C^r manifold in $\mathbb{R}^n$, $\dim M = m$, M - compact, $r \geq 1$

Lemma 1: Let $U \subseteq \mathbb{R}^m$ open, $f: U \xrightarrow{C^r} \mathbb{R}^s$, $K \subset U$ compact. Then $\forall \varepsilon > 0$ $\exists g: \mathbb{R}^m \to \mathbb{R}^s$, $g \in C^\infty$ s.t. $\|(f-g)|_K\|_{C^r} < \varepsilon$

(Proof: g - a suitable convolution of f and some C^∞ function)

Corollary 2: $C^\infty(M, \mathbb{R}^s)$ are dense in $C^r(M, \mathbb{R}^s)$

Proof: Take a C^∞ partition (rozkład jedności) of unity on M $\{\varphi_j\}$ w.r.t. $\{V_j\}$ i.e.

$\{V_j\}_{j=1}^k$ C^r atlas on M

$$\varphi_j : M \xrightarrow{C^\infty} \mathbb{R}$$

$$\varphi_j \equiv 0 \quad \text{outside } V_j$$

$$\sum_{j=1}^k \varphi_j(x) = 1 \quad \forall x \in M$$

Let $f: M \xrightarrow{C^r} \mathbb{R}^s$

then ~~Let~~ $f \circ \psi_j : W_j \xrightarrow{C^r} \mathbb{R}^s$

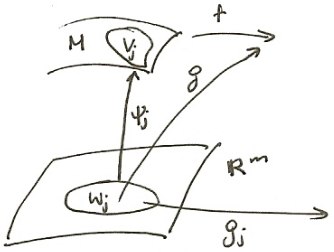
By lemma 1 $\exists g_j : \mathbb{R}^m \xrightarrow{C^\infty} \mathbb{R}^s$ for arbitrary small δ

$\|f \circ \psi_j - g_j\| < \delta$

$\|\varphi_j \cdot f \circ \psi_j - \varphi_j \cdot g_j\| < \frac{\varepsilon}{k}$

let $g = \sum_{j=1}^k \varphi_j g_j \circ \psi_j^{-1}$

$f = \underbrace{\sum_{j=1}^\infty \varphi_j}_{1} f$



then $\|f-g\|_{C^r} = \|\sum_j \varphi_j \cdot f - \sum_j \varphi_j g_j \cdot \psi_j^{-1}\|_{C^r} \leq \sum_j \|\varphi_j f - \varphi_j g_j \circ \psi_j^{-1}\|_{C_r} \leqslant \sum_j \frac{\varepsilon}{k} = \varepsilon$

Lemma 3: $\mathrm{Diff}^r(M)$ is open in $C^r(M,M)$ (in C^r topology)

Proof: Let $f \in \mathrm{Diff}^r(M)$, $x \in M$. By local invertible theorem if g is suff. C^r-close to f then g is 1-1 map in a neighbourhood of x. Then we choose a finite cover of M by such neighbourhoods etc ...

Def: $f: M \xrightarrow{C^r} M$ diff, $x \in M$, then ω-limit set of x, $\omega(x)$ is set of all limit points of the sequence $(f^n(x))_{n=1}^\infty$

$y \in \omega(x) \iff \exists n_k \to \infty \quad f^{n_k}(x) \xrightarrow[k\to\infty]{} y$

$(y \in \alpha(x) \iff \exists n_k \to -\infty \quad f^{n_k}(x) \xrightarrow[k\to\infty]{} y \quad)$

V- C^r vector field on M $\{\varphi_t\}$ flow

$y \in \omega(x) \iff \exists t_n \to +\infty \quad \varphi_{t_n}(x) \xrightarrow[n\to\infty]{} y$

$y \in \alpha(x) \iff \exists t_n \to -\infty$ — " —

Fact: $\forall\ x \in M$ $\quad w(x)$ is non-empty compact, invariant

For flows, if M connected then $w(x)$ connected

Proof:

(a) $w(x) \neq \emptyset$ $\quad \{f^n(x)\}$ a sequence, M compact, so $\exists\ n_k$ s.t. $f^{n_k}(x) \to y$

$\varphi_n(x)$ — " ———— $\varphi_{n_k}(x) \to y$

(b) $w(x)$ is invariant

$$f^{n_k}(x) \to y \quad \Downarrow \quad f^{n_k+1}(x) \to f(y)$$

$$\varphi_{t_n}(x) \to y \quad \Downarrow \quad \varphi_{t+t_n} = \varphi_t(\varphi_{t_n}(x) \to \varphi_t(y))$$

(c) $w(x)$ is compact

$y_n \to y$ $\quad y_n \in w(x)$ $\quad \varphi_{t_n^j}(x) \to y$

$\varepsilon > 0$ choose n such that $\|y_n - y\| < \frac{\varepsilon}{2}$

Choose j s.t. $\|y_n - \varphi_{t_n^j}(x)\| < \frac{\varepsilon}{2}$ and $t_n^j > n$

then $\|\varphi_{t_n^j}(x) - y\| < \varepsilon$, $\quad y \in w(x)$

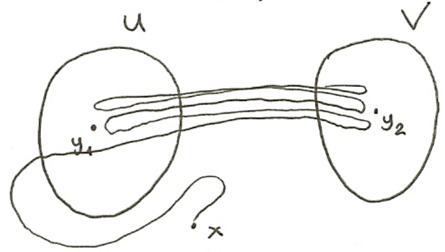
(d) Asoume that $w(x)$ is not compact

Then, since $w(x)$ compact

$\exists\ U, V$ open in M $\quad U \cap V = \emptyset$, $\quad U \cap w(x) \neq \emptyset$

$V \cap w(x) \neq \emptyset$

$w(x) \subset U \cup V$ $\quad \bar{U} \cap \bar{V} = \emptyset$



$y_1, y_2 \in w(x)$

By definition of $w(x)$ $\quad \exists\ t_n \to +\infty$ $\quad \varphi_{t_n}(x) \to y_1$

$t_n' \to +\infty$ $\quad \varphi_{t_n'}(x) \to y_2$

taking a subsequence of t_n, t_n' we can assume

$\varphi_{t_n}(x) \in U$

$\varphi_{t_n'}(x) \in V$

$$\ldots < t_n < t_n' < t_{n+1} < t_{n+1}' < \ldots$$

$\exists\ s_n \in (t_n, t_n')$ $\quad \varphi_{s_n}(x) \in \partial U$

∂U compact, so

$\varphi_{s_{n_j}}(x) \to y \in \partial U$

$y \in w(x)$ but ∂U is disjoint with $w(x)$. contradiction

<u>Def:</u> we say that a point $x \in M$ is wondering ([illegible]) <u>for flows</u>

if $\exists$ neighbourhood U of x and t_0 s.t. $\varphi_t(U) \cap U = \emptyset$ $\quad \forall\ t > t_0$

for diff $\exists$ U- neighbourhood of x and n_0 s.t.

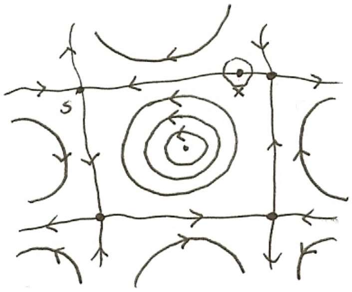
$f^n(U) \cap U = \emptyset$ $\forall\ n > n_0$

$\Omega(V)$ the set of non-wandering points

Def: $x \in M$ is recurrent if $x \in \omega(x)$

Fact: x is reccurent $\Rightarrow$ x is non-wandering (obvious)

Ex:



$\omega(x) = \{s\}$

if U is neighbourhood of x, then a part of U behaves under φ_t like under a rotation and returns ∞- many times to itself so $x \in \Omega$

Fact: $\Omega(V)$ is non- empty, invariant, compact, connected (if M connected)

V- C^r vecta field and $\Omega(V) \supset \omega(x)$ $\forall\ x \in M$

$y \in \omega(x)$

U- nghb of y

fixed n

m - larger $m \to \infty$

$$\varphi_{t_m - t_n}(U) \to \varphi_{t_m - t_n}(\varphi_{t_m}(x)) = \varphi_{t_m}(x) \in \mathcal{U}$$

$$\varphi_{t_m - t_n}(U) \cap U \neq \emptyset$$

$S_m = t_m - t_n$ $S_m \to +\infty$

then $\varphi_{s_n}(U) \cap U = \emptyset$

$\varphi_t(U) \cap \varphi_{t_n + t}(U) = \emptyset$

8.04. 2011

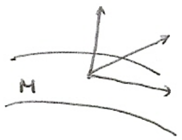
Gradient vector fields (pola gradientowe)

M- compact, C^r - manifold

$M \subset \mathbb{R}^n$, a vector field $V: M \to TM$ is a gradient field if

$V = \underset{\substack{\uparrow \\ \text{comp. of grad } f \\ \text{tangent to } M}}{\text{grad } f}$, where $f: \mathcal{U} \to \mathbb{R}$, $f \in C^{r+1}$

$\mathcal{U}$- neighbourhood of M, $\mathcal{U} \subset \mathbb{R}^n$



Generally, if $f: M \to \mathbb{R}$, $f \in C^{r+1}$, then $\forall x \in M$

$Df(x)(h) = \langle v(x), h \rangle$ for some $v(x)$

Then we call V a gradient vector field

Fact: If V is a gradient field, then it has no (non- trivial) closed orbits

Lemma: The function f increases along trajectories of the gradient flow φ_t.

Proof:

$$\frac{d}{dt} f(\varphi_t(x)) = (\nabla f)\Big(\underbrace{\frac{\partial}{\partial t}\varphi_t(x)}_{\overset{\shortparallel}{v(\varphi_t(x))}}\Big) = |\nabla f|^2_{\varphi_t(x)} \geqslant 0$$

$|\nabla f|^2 > 0$ unless $\nabla f(\varphi_t(x)) = 0 \iff v(\varphi_t(x)) = 0$

Fact 2: If V is a gradient vector field, then $\forall x \in M$ $\omega(x)$ consists of singuler points of V.

Proof: Suppose that $y \in \omega(x)$, y - regular point of V.

$S = f^{-1}(f(y))$ $\quad S = \{ z : f(z) = f(y) \}$

by the implicit function theorem, S is a $(n-1)$ dimensional manifold in a neighbourhood of y. such that $V(z)$ is ~~a~~ normal to S in a neighbourhood of y

$y \in \omega(x)$ so there ~~are~~ is a sequence $t_n \to +\infty$ s.t. $\varphi_{t_n}(x) \in S$

Then points of trajectory of x close to $\varphi_{t_n}(x)$ are $\overset{\approx y}{\text{normal}}$ to manifold $f \equiv$ const, so ~~f increases along all these points~~ $\exists$ t_n' s.t.

$\varphi_{t_n'}(x) \in S$

$\varphi_{t_n'}(x) \longrightarrow y$

But this is impossible since $f(\varphi_{t_n'}(x)) = c$ and f increases along the trajectories.

Fact: $A \in GL(\mathbb{R}^n)$ hyperbolic isomorphism of $\mathbb{R}^n$. (eigenvalues have modulus $\neq 1$)

$$\mathbb{R}^n = \underset{\substack{\uparrow \\ \text{stable}}}{E^s} \oplus \underset{\substack{\uparrow \\ \text{unstable}}}{E^u}$$

Then there exists a norm $\|\cdot\|$ in $\mathbb{R}^n$ s.t. $\|A^s\|_1 < 1$, $\|(A^u)^{-1}\| < 1$, where $A^s = A|_{E^s}$, $A^u = A|_{E^u}$

Proof: There exists a linear basis in $\mathbb{R}^n$ s.t. the matrix of A in this basis has a Jordan form

Consider A^s, E^s

$$M = \begin{bmatrix} \begin{bmatrix} \lambda_i & & \\ 1 & \ddots & \\ & 1 & \lambda_i \end{bmatrix} & & & \\ & \ddots & & \\ & & \begin{matrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \\ -1 & 0 \\ 0 & 1 \end{matrix} & \\ & & & \ddots \end{bmatrix}$$

$|\lambda_i| < 1 \quad \forall i j$

$\alpha_{ij}^2 + \beta_j^2 < 1$

let $M(t)$ be the matrix M, where all "1" under the diagonal are replaced by "t", $t \in \mathbb{R} \setminus \{0\}$.

Then there exists a basis $\{v_1(t), \dots, v_s(t)\}$ A^s has matrix $M(t)$ in this basis.

(because $N(t) = \begin{bmatrix} t & & & 0 \\ & t^2 & & \\ & & \ddots & \\ 0 & & & t^s \end{bmatrix} M \begin{bmatrix} 1/t & & & 0 \\ & \frac{1}{t^2} & & \\ & & \ddots & \\ 0 & & & 1/t^s \end{bmatrix}$)

$t y_1 = y_1'$

$t x_1 = t x_1'$

$y_1 = \lambda x_1$

$y_2 = x_1 + \lambda x_2$

$y_3 = x_2 + \lambda x_3$

$\vdots$

$t y_1 = \lambda t x_1$

$t^2 y_2 = t^3 x_1 + t^2 \lambda x_2$

Let $A(t)$ - linear map of matrix $M(t)$

$\| A(0) \| = \max\limits_{i,j} \left(|\lambda_i|, \sqrt{\alpha_j^2 + \beta_j^2} \right) < 1$

$\| A(t_0) \| < 1$ for sufficiently small t_0

Define $\| \cdot \|_1$ as the norm given by scalar product $\langle \cdot, \cdot \rangle_1$

s.t. $\langle v_i(t_0), v_j(t_0) \rangle = \delta_{ij}$, then

$\| A^s \|_1 = \| A(t_0) \| < 1$

<u>fact:</u> Two hyperbolic linear vector fields V, W in $\mathbb{R}^n$ have conjugated flows $\Leftrightarrow$ the dimension of E^s, E^u are the same for V and W.

Conjugation

$\exists h \quad \underset{\substack{| \\ \text{flow for } V}}{h \circ \varphi_t} = \underset{\text{flow for } W}{\psi_t \circ h}$

V_t

$\dot{x} = \underset{\substack{| \\ \text{matrix of hyperb. isom.}}}{M} x$

<u>Proof:</u> Let E^s, E^u be the stable and unstable space for V

E'^s, E'^u ——— '' ——— W

$\Rightarrow \quad \begin{matrix} h(E^s) = E'^s \\ h(E^u) = E'^u \end{matrix} \quad \Rightarrow \quad \begin{matrix} \dim E^s = \dim E'^s \\ E^u = E'^u \end{matrix}$

h - homeo

$\Leftarrow$ construction of h

we define h s.t. $h(x) = h^s(x_s) + h^u(x_u)$

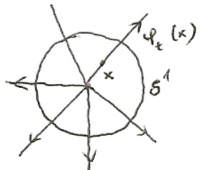
$x = \underset{\substack{\cap \\ E^s}}{x^s} + \underset{\substack{\cap \\ E^u}}{x^u}$

$h^s : E^s \to E'^s$

$h^u : E^u \to E'^u$

homeo

Construction of h^u:



It is enough to assume $V(x) = x$

Let $\tau(x)$ = time of arriving at S^1

$\tau(x)$ is the unique t s.t. $\varphi_t(x) \in S^1$

Let $h^u(x) = \psi_{-\tau(x)}(\varphi_{\tau(x)}(x))$ $x \neq 0$

$\mathbb{R}^n$

15.04.2011

Fact: V, W - hyperbolic vector fields in $\mathbb{R}$

φ_t flow defined by V

ψ_t flow defined by W

Then φ_t is conjugate to ψ_t. $\exists$ homeo $h: \mathbb{R}^n \circlearrowleft$ $h \circ \varphi_t = \psi_t \circ h$

$\Longleftrightarrow$ $\dim E^u = \dim E'^u$

$\dim E^s = \dim E'^s$

where $\mathbb{R}^n = E^s \oplus E^u$

$\mathbb{R}^n = E'^s \oplus E'^u$

Proof:

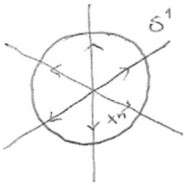
$\Leftarrow$ obvious

$\Rightarrow$ $h(x) = h(x_s + x_u) = h^s(x_s) + h^u(x^u)$

$\underset{\underset{E^s}{\Uparrow}}{x = x_s} + \underset{\underset{E^u}{\Uparrow}}{x_u}$, $\underset{\text{homeo}}{h^s: E^s \to E'^s}$, $\underset{\text{homeo}}{h^u: E^u \to E'^u}$

construction of h^u: assuming that $V|_{E^u}(x_u) = x_u$

$$\begin{cases} h^u(x_u) = \psi_{-\tau(x_n)} \circ \varphi_{\tau(x_n)}(x_n) & x_n \neq 0 \\ h^u(0) = 0 \end{cases}$$



$\tau(x)$ the time of reaching S^1

$\varphi_{\tau(x)}(x^n) \in S^1$

h^u is homeomorphism

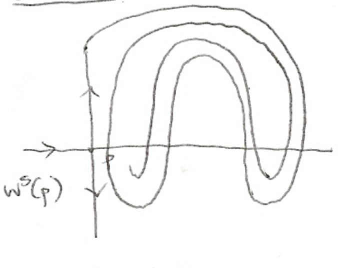
$h \circ \varphi_t(x) \overset{?}{=} \psi_t \circ h(x)$

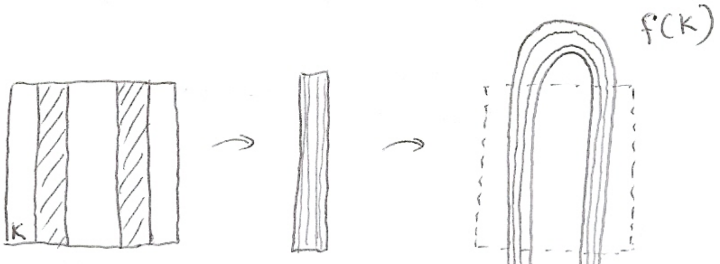
$L = \psi_{-\tau(\varphi_t(x))} \circ \varphi_{\tau(\varphi_t(x))} \circ \varphi_t(x) = \psi_{-\tau(\varphi_t(x))} \circ \varphi_{\tau(\varphi_t(x))+t} =$

$= \psi_{-\tau(\varphi_t(x))} \circ \varphi_{\tau(x)}(x)$

$R = \psi_t \circ \psi_{-\tau(x)} \circ \varphi_{\tau(x)}(x) = \psi_{t-\tau(x)} \circ \varphi_{\tau(x)}(x) = L$

Smale's horseshoe:

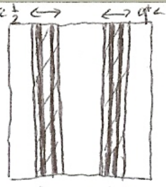


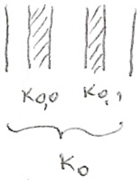


$f: K \to \mathbb{R}^2$

p hyperbolic

$f(K) \cap K = K_0^+ \cup K_1^+$





$K_{i_1 \dots i_{n+1}} \subset K_{i_1 \dots i_n}$

$f^2(K) \cap f(K) \cap K$

$= f^2(K) \cap (K_0 \cup K_1)$

$= f(K_0 \cup K_1) \cap (K_0 \cup K_1) = K_{0,0}^+ \cup K_{0,1}^+ \cup K_{1,0}^+ \cup K_{1,1}^+$

$f^n(K) \cap f^{n-1}(K) \cap \dots \cap K = \bigcup_{i_1 \dots i_n} K_{i_1 \dots i_n}$ ← rectangles of width $(q^+)^n$

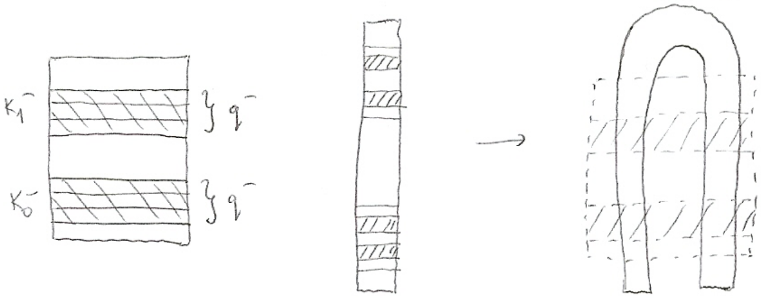
$i_j \in \{0,1\}$

$\bigcup_{i_1 \dots i_n} K_{i_1 \dots K_{i_n}}^+ = \{x \in K : f^{-k}(x) \in K \quad \forall_{k=1,\dots,n}\}$

$f^{-1}(K) \cap K = K_0^- \cup K_1^-$

$f^{-n}(K) \cap f^{-(n-1)}(K) \cap \dots \cap K$

$= \bigcup_{i_1 \dots i_n} K_{i_1, \dots, i_n}^- =$



↑

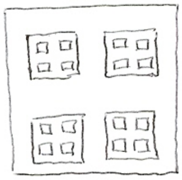
„horizontal" rectangles of height $(q^-)^n$, $q^- < 1$.

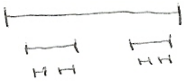
$= \{x \in K : f^{-n}(x) \in K \quad \forall n \in \mathbb{N}\} = C^+ \times [0,1]$

a cantor set

$\{x \in K : f^{-n}(x) \in K \quad \forall n \in \mathbb{N}\} = C^+ \times [0,1]$

a Cantor set

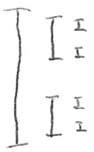


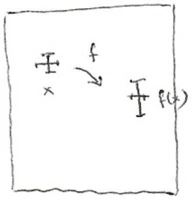


a Cantor set

$\{x \in K, \ f^n(x) \in K \quad \forall n \in \mathbb{N}\} = [0,1] \times C^-$

$\{x \in K : \ f^n(x) \in K \quad \forall n \in \mathbb{Z}\} = C^+ \times C^-$





$x \in C^+ \times C^-$

$Df(x)$ hyperbolic (with horizontal attraction and vertical expansion)

$$T_x\mathbb{R}^2 \simeq \mathbb{R}^2 = \underset{E^s}{\underset{\parallel}{\mathbb{R}}} \oplus \underset{E^u}{\underset{\parallel}{\mathbb{R}}}$$

$\Lambda = C^+ \times C^-$ is hyperbolic

Each point $x \in \Lambda$ has a symbolic coding (itinerary)

$n \geq 0$ $\qquad$ $n < 0$

$f^n(x) \in K_0^+ \cup K_-^+$ $\qquad$ $f^n(x) \in K_0^- \cup K_1^-$

Let $\cancel{f^n} \, i_n(x) = \begin{cases} 0 & \text{when } f^n(x) \in K_0^+ \\ 1 & \text{-"- } f^n(x) \in K_1^- \end{cases}$ $\qquad$ $i_n(x) = \begin{cases} 0 & \text{when } f^n(x) \in K_0^- \\ 1 & \text{- " - } K_1^- \end{cases}$

$x \longmapsto (\dots i_n, \dots, i_{-1}, i_0, i_1, \dots, i_n \dots)$

1-1 $\qquad \forall j \; i_j = 0,1$

$\Sigma_2 = \{0,1\}^{\mathbb{Z}} \qquad \Sigma_2^+ = \{0,1\}^{\mathbb{N}}$

Σ_2 has standard topology of a cartesian product (Σ_2 compact)

metric

$$\rho((i_n),(i_n')) = \frac{1}{2^{|k|}} \quad \text{where } |k| \text{ is minimal s.t. } i_k \neq i_k'$$

$$= \sum_{n\in\mathbb{Z}} \frac{|i_n - i_n'|}{2^{|n|}}$$

$\sigma : \Sigma_2 \to \Sigma_2$ left-side shift $\sigma((i_n)_{n\in\mathbb{Z}}) = ((i_{n+1})_{n\in\mathbb{Z}})$

$\Sigma_2^+ \to \Sigma_2^+$

The same for

$\Sigma_k = \{0,\dots,k-1\}^{\mathbb{Z}}$

(Σ_k, σ) full k-shift

$\sigma : \Sigma_k \to \Sigma_k$ - homeo

$\sigma : \Sigma_k^+ \to \Sigma_k^+$ k to 1 map

topological Markov chain

Fact: $f \circ \pi = \pi \circ \sigma$

by deffinition

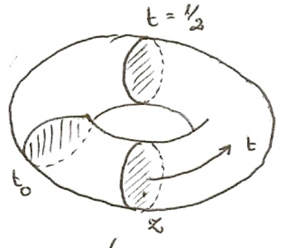
Remark: If we perturb (slightly) f in C^1, the f still will be conjugate to π.

Corollary:

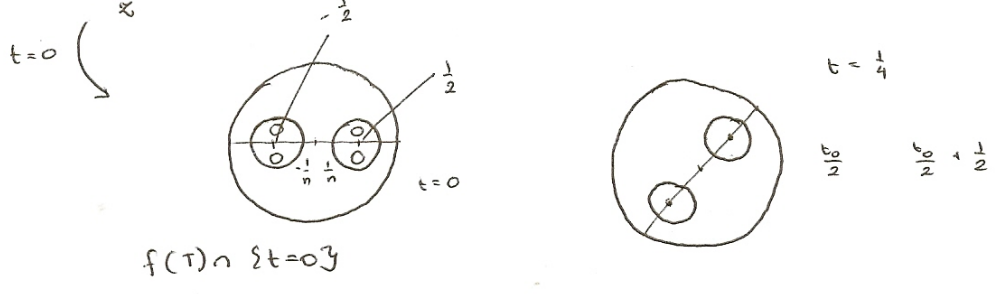
$f|\Lambda$ is C^1 structurally stable.

29. 04. 2011

Example : Consider a fat torus $S^1 \times D^2$

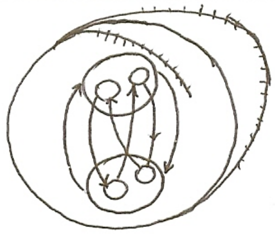


$$f(t,z) = (2t \,(\text{mod } 1), \tfrac{1}{4}z + \tfrac{1}{2}e^{2\pi i t})$$



$f(T) \cap \{t=0\}$

$f(T) \subset \text{int } T \qquad \bigcap_{n\geq 0} f^n(T) = \Lambda$ - hyperbolic set, attractor



$$Df(t,z) = \begin{pmatrix} 2 & 0 \\ \pi i e^{2\pi i t} & 1/4 \end{pmatrix}$$

Invariant cones:

check that the family of cones $(u,v) : |u| > c\|v\|$ is invariant, for some c

$$\overset{U}{} \qquad \overset{V}{}$$

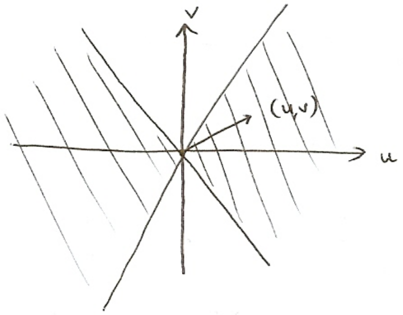
$$\begin{pmatrix} 2 & 0 \\ \pi i e^{2\pi i t} & 1/4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = (2u, \pi i e^{2\pi i t} u + \tfrac{1}{4} v)$$

$|u| > c\|v\|$

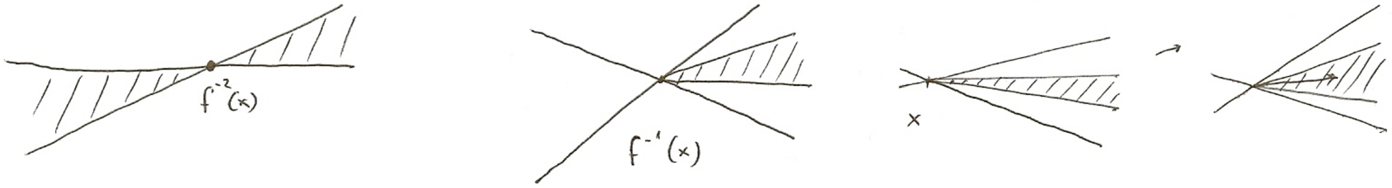
$\Downarrow$

$|U| > c\|V\|$

$c = 2$ should be ok



$\| Df^n((u,v)) \| \geq \tilde{\lambda}^n \|(u,v)\|$



C_x - cone invariant detected by us. we want to check that

$$\bigcap Df^n(C_{f^{-n}x}) = E^u_x \quad \text{our unstable direction}$$

$(u, v_1) \in C$

$(u, v_2) \in C$

$(U, V_1) = Df(u, v_1)$

$(U, V_2) = Df(u, v_2)$

Difference of slopes will become smaller. We're interested in

$$\left(\frac{\|V_1\|}{\|U\|} - \frac{\|V_2\|}{\|U\|} \right)$$

$$\left(\frac{\|v_1\|}{\|u\|} - \frac{\|v_2\|}{\|u\|} \right)$$

$$\left| \|\pi i e^{2\pi i t} u + \tfrac{1}{2} v_1\| - \|\pi i e^{2\pi i t} u + \tfrac{1}{2} v_2\| \right| \leq c \left| \|v_1\| - \|v_2\| \right|$$

we want to $\dfrac{\|V_1 - V_2\|}{\|U\|} = \dfrac{1}{8} \dfrac{\|v_1 - v_2\|}{|u|}$

So Λ is a hyperbolic attractor (Λ is called Solenoid or Smale attractor)

There is a fixed point (easy).
There are many periodic points, (we want to check it)
How to find periodic orbit of period two ?

20.05.2011

Birkhoff ergodic theorem $\qquad (X, \mathcal{M}, \mu)$ measure space $\mu(X) = 1$

$\forall\ f : X \to \mathbb{R}\ ,\ f \in L_1(\mu) \qquad T : X \to X \quad \mu$- invariant

$$\frac{f(x) + f(Tx) + \ldots + f(T^{n-1}x)}{n} \underset{\substack{a.e. \\ L^1}}{\longrightarrow} f \qquad f = \mathbb{E}(f \mid \mathcal{G}) \qquad \mathcal{G} = \sigma\{A : f^{-1}(A) \underset{a.e.}{=} A\}$$

Def: μ is ergodic if $\forall\ A \in \mathcal{M} \quad T^{-1}(A) = A \Rightarrow \mu(A) = 0$ or 1.

Theorem: The following are equivalent

(a) μ is ergodic

(b) $\forall A \in \mathcal{M} \quad \mu(T^{-1}(A) \div A) = 0 \Rightarrow \mu(A) = 0$ or 1

(c) $\forall f \in L^1 \quad \dfrac{f(x) + \ldots + f(T^{n-1})}{n} \underset{a.e.}{\longrightarrow} \displaystyle\int_X f\, d\mu$

(d) $\forall f \in L^1$ if $f(T(x)) \underset{a.e.}{=} f(x)$ (f is T- invariant) then $f \underset{a.e.}{=} \text{const}$

(e) $\forall f \in L^1$ if $f(T(x)) \underset{(\leq)}{\geq} f(x)$ then $f \underset{a.e.}{=} \text{const}$

(f) $\forall A \in \mathcal{M} \quad \mu(A) > 0 \quad \mu\left(\bigcup_{n=0}^{\infty} T^{-n}(A)\right) = 1$

(g) $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}(A) \cap B) = \mu(A)\mu(B) \quad \forall\ A, B \in \mathcal{M}$

(h) μ is an extremal point of the set $\mathcal{M}$ of all T- invariant probabilistic measures on X

Def: Let Y be a convex set. A point $x \in Y$ is extremal if $x = ty + (1-t)z \quad y, z \in V \Rightarrow y = z = x$ or $t = 0, 1$
$0 \leq t \leq 1$

Proof: (a) $\Rightarrow$ (b)
(b) If $\mu(A \div T^{-1}(A)) \overset{\pm 101}{=} 0$ Then for $B = \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} T^{-k}(A)$ we have $B = T^{-1}(B)$ and $\mu(B \div A) = 0$.
from (a) $\mu(B) = 0$ or 1 so $\mu(A) = 0$ or 1

(a) $\Rightarrow$ (c) from Ergodic theorem

(c) $\Rightarrow$ (a) $\quad f = \chi_A$
suppose $T^{-1}(A) = A$ then for $f = \chi_A$
$f(T^k(x)) = \chi_A(T^k x) = \chi_{T^{-k}(A)}(x) = \chi_A(x) = f(x)$

$$\int f \longleftarrow \frac{f(x) + \ldots + f(T^{n-1}x)}{n} = \frac{nf(x)}{n} = f(x) \Rightarrow f = \text{const a.e} \Rightarrow \mu(A) = 0 \text{ or } 1$$

(c) => (e)

If $f(Tx) \geq f(x)$ then $\frac{f(x) + \dots + f(T^{n-1}x)}{n} \geq f(x)$

$$\swarrow$$

$$\int f \, d\mu$$

$$f \leq \int f \, d\mu \quad \text{a.e} \quad \Rightarrow \quad f \underset{a.e}{=} \int f \, d\mu$$

<u>Fact:</u> (a) If μ is T-invariant, ergodic measure, ν T-invariant measure $\nu << \mu$ (ν absoluetly continuous with respect to μ), then $\mu = \nu$

$\mu(A) = 0 \Rightarrow \nu(A) = 0$

(b) If μ, ν are T- invariant measures and ergodic then $\mu = \nu$ or $\mu \perp \nu$ (μ, ν are mutually singular $\exists$ A, B s.t. $\mu(A) = 0, \mu(B) = 0, \nu(A) = 1, \nu(B) = 1$)

<u>Proof:</u>

(a) Take $A \in \mathcal{M}$. By ergodic thm for μ

$$\frac{1}{n}\left(\chi_A(x) + \dots + \chi_{T^{-(n-1)}A}(x)\right) \xrightarrow[\mu \text{ a.e.}]{} \mu(A) \quad (*)$$

$$\chi_{T^{-(n-1)}A}(x) = \chi_A(T^{n-1}x)$$

by absoluetly continuity of ν with respect to ν

(*) holds also for ν - a.e x

$$\int \lim_n \left(\frac{1}{n} \chi_A(x) + \dots + \chi_{T^{-(n-1)}A}(x)\right) d\nu = \mu(A)$$

|| Lebesgue thm

$$\lim_n \int \text{ — || — }$$

$$\| $$

$$\frac{1}{n}\left(\nu(A) + \dots + \nu(T^{-(n-1)}A)\right) = \mu(A)$$

$$\| $$

$$\nu(A)$$

so $\nu(A) = \mu(A)$

let $A \in \mathcal{M}$

(b) $B = \{x : \frac{1}{n}\left(\chi_A(x) + \dots + \chi_A(T^{n-1}x)\right) \xrightarrow{n \to \infty} \mu(A)\}$

$C = \{x : \frac{1}{n}\left(\chi_A(x) + \dots + \chi_A(T^{n-1}x)\right) \xrightarrow[n \to \infty]{} \nu(A)\}$

By ergodic theorem $\mu(B) = 1, \nu(C) = 1$ ~~$1 > \nu(A) > 0$~~

If $\mu \neq \nu$ then ~~$\exists A$ s.t. $1 > \mu(A) > 0$~~

$B \cap C \neq \emptyset$

take $x \in B \cap C$

then $\frac{1}{n}\left(\chi_A(x) + \dots + \chi_A(T^{n-1}x)\right) \to \mu(A)$

$\to \nu(A)$, $\mu(A) = \nu(A)$

<u>Corolary:</u> the point (h) is equivalent to ergodicity

<u>Proof:</u> (a) => (h) suppose μ is ergodic and is not an extremal point of $\mathcal{M}$. Then $\mu = t\mu_1 + (1-t)\mu_2$ $\quad \mu_1, \mu_2 \in \mathcal{M}$, $\mu_1 << \mu$

$0 < t < 1$

so $\mu_1 = \mu$

(h) $\Rightarrow$ (a) If μ is not ergodic, then $\exists A$ $T^{-1}(A) = A$

then $\mu = \mu(A)\mu_1 + \mu(X\setminus A)\mu_2$ $\quad \mu(A) \neq 0, 1$

$$\mu_1(B) = \frac{\mu(B\cap A)}{\mu(A)} \qquad \mu_2(B) = \frac{\mu(B\cap(X\setminus A))}{\mu(X\setminus A)}$$

so μ is not extremal

Corollary: (of Krylov - Bogoljubov theorem)

X - compact, T continuous then $\exists$ T-invariant ergodic measure.

Proof:

A measure is non-negative functional in $C^*(X)$

$(\mu(f) := \int f\, d\mu)$

~~{probabilistic measures} $\subset C^*(X)$~~

compact, ~~convex set (weakly *)~~ $\mathcal{M} \subset C^*(X)$

$\mathcal{M} \neq \emptyset$ by Kirylov - Boguljubov thm

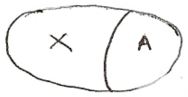
$\mathcal{M}$ compact, convex so by Krein - Milman thm

$\exists$ extremal point in $\mathcal{M}$ (and they are ergodic measures)

Kac lemma (1947)

If μ is ergodic then

$$\int_A n_A(x)\, d\mu(x) = 1$$



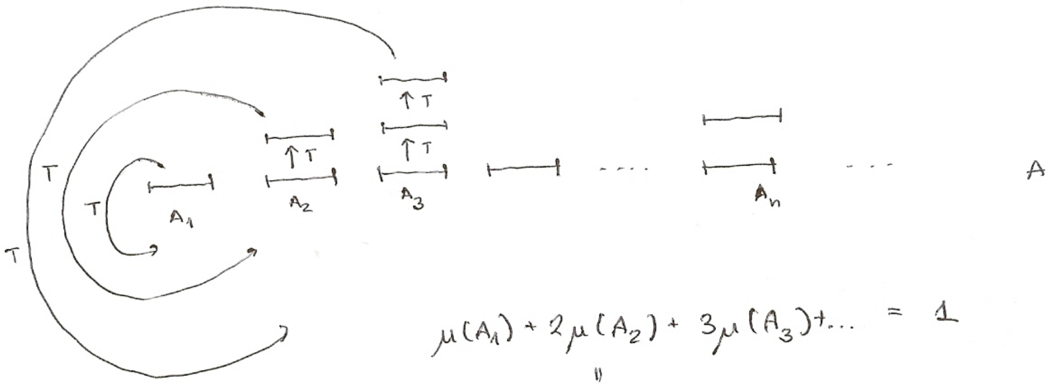
$\mu(A) > 0$

$n_A(x) = \inf\{n \geq 1 : T^n x \in A\}$
$x \in A$

$$\sum_{n=1}^{\infty} n \mu A_n = 1$$

$A_n = \{x : n_A = n\}$

Proof: (Assume T - automorphism) $\mu(T(A)) = \mu(A)$



$$\mu(A_1) + 2\mu(A_2) + 3\mu(A_3) + \ldots = 1$$
$$\|$$
$$\int n A\, d\mu$$

27.05.2011

⌂ from 2 weeks ago

Check that Lebesgue measure on $S^1 \subset \mathbb{C}$ is preserved under Blaschke products of the form

$$B(z) = \prod_{k=1}^{n} e^{it_k} \frac{z-a_k}{1-\bar{a}_k z} \qquad \begin{matrix} a_k \in \mathbb{C} \\ |a_k| < 1 \\ t_k \in \mathbb{R} \end{matrix}$$

provided $a_k = 0$ for some k $\iff$ $B(0) = 0$

Fact: T preserves μ $\iff$ $\forall f \in L^1$ $\int f d\mu = \int f \circ T d\mu$

so : $\mu = Leb_{S_1} = L$ $\quad \int_{S_1} f dL \overset{(*)}{=} \int_{S_1} f\circ B \, dL$

1) It is sufficient to prove it for $f \in C^0$ (standard)

2) To prove it for $f \in C^0$ it is sufficient to prove it for polynomials and $\overline{\text{polynomials}}$ ← complex conjugate

$$\int_{S_1} P(B(z)) dL = \int_0^{2\pi} P(B(e^{it})) dt = 2\pi \, P(B(0)) = 2\pi \, P(0) = \int_0^{2\pi} P(e^{it}) dt =$$
$$= \int_{S_1} P dL$$

Ex 2: $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$

A - integer matrix $n \times n$

$T(x)$ is well defined on $\mathbb{T}^n$

$T: \mathbb{T}^n \to \mathbb{T}^2$, $\det A = \pm 1$

Then T preserves Leb measure on $\mathbb{T}^n$

we have to check that T is invertible on $\mathbb{T}^n$.

so $T/\mathbb{Z}^n$ is invertible

by

so $0 \in \mathbb{T}^n$ has exactly one preimage in $\mathbb{T}^n$

$\Rightarrow \forall x \in \mathbb{T}^n$ has only one preimage

More generally if $|\det A| = d$ then $\det T = d$ on $\mathbb{T}^n$

Examples of ergodicity:

1. Rotation on the circle of irrational angle

$T(z) = e^{2\pi i \alpha} z$, $\alpha \notin \mathbb{Q}$

$|z| = 1$

$T(t) = t + \alpha \mod 1$ $\quad t \in \mathbb{R}$

Formulation of the problem :

$$\frac{\# \{ k \le n : T^k(0) \in [a;b] \}}{n} \overset{?}{\to} b-a$$

Proof: Lebesgue density theorem:

If $A \subset \mathbb{R}^n$ is Lebesgue measurable and $Leb(A) > 0$ then for almost every point $x \in A$

$$\lim_{r \to 0^+} \frac{Leb(A \cap B(x,r))}{Leb(B(x,r)} = 1$$ (x is called a density point)

Let $A \subset S^1$, $T(A)=A$ we'll show that $Leb(A)=0$ or 1. Suppose not then $Leb\ A > 0$, $Leb\ (S^1 \setminus A) > 0$.

let x be a density point for A, y be a density point for $S^1 \setminus A$.

Take r s.t. $Leb\ (A \cap B(x,r)) \geq \frac{99}{100}\ Leb\ B(x,r)$

$Leb\ (S^1 \setminus A \cap B(y, 2r)) > \frac{99}{100}\ Leb\ (y, 2r)$

n s.t. $\|T^n x - y\| \leq \frac{r}{10}$, then $T^n(B(x,r)) \subset B(y, 2r)$,

so $Leb\ (A \cap B(y, 2r)) > \frac{99}{100}\ Leb\ (y, 2r)$ impossible.

$T: \mathbb{T}^n \to \mathbb{T}^n$, $T(x_1, \dots, x_n) = (x_1 + \alpha_1, \dots, x_n + \alpha_n)$ mod 1

$x_k \in \mathbb{R}/\mathbb{Z}$ $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ (rotation in every direction)

if $1, \alpha_1, \dots, \alpha_n$ are lineary independent on $\mathbb{Q}$ $\Leftrightarrow$ $k_1\alpha_1 + \dots + k_n\alpha_n = n$, $k \in \mathbb{Z}, n \in \mathbb{Z}$ $\Rightarrow$ $\forall_j\ k_j = 0$.

Let $T: \mathbb{T}^n \to \mathbb{T}^n$ be an algebraic automorphism of $\mathbb{T}^n$ of matrix A then T is ergodic $\Leftrightarrow$ A has no eigenvalue which are roots of unity.

Proof: we prove that if $f \in L^1$ is T-invariant (ie $f \circ T = f$ a.e) then f is constant a.e. It is sufficient to show it for $f \in L^2$.

suppose $f \circ T = f$. we know that $f(x) = \sum_{s \in \mathbb{Z}^n} c_s e^{2\pi i \langle s,x \rangle}$

$$f \circ T(x) = \sum c_s e^{2\pi i \langle sAx \rangle} = \sum c_s e^{2\pi i \langle A^* s, x \rangle}$$

$$f(x) = \sum_{s \in \mathbb{Z}^n} c_s e^{2\pi i \langle s,x \rangle} \underset{\uparrow}{=} \sum_{s \in \mathbb{Z}^n} c_{A^* s} e^{2\pi i \langle A^* s, x \rangle} = \sum c_s e^{2\pi i \langle A^* s, x \rangle}$$

A^* - automorphism of $\mathbb{Z}^n$

By the uniqueness of Fourier coefficient $c_s = c_{A^* s}$ $\forall\ s \in \mathbb{Z}^n$

$c_s = c_{A^* s} = c_{(A^*)^2 s}$

suppose $\exists\ k \in \mathbb{N}$, $s \neq 0$, $s \in \mathbb{Z}^n$ $(A^* s)^k = s$

$(A^*)^k$ on the linear subspace $\underbrace{span(s, A^* s, \dots, (A^* s)^k)}_{S}$ is identity

A^* on S has an eigenvalue λ

$\lambda^k = 1$ but this contradicts the assumption $(A^* s)^k \neq s$

so for every k and every $s \neq 0$, $s \in \mathbb{Z}^n$

so all indexes are diffrent, so coefficiants must be 0, because

$f \in L_2$: so f is constant a.e

$\Rightarrow$ if $\exists\ \lambda$: that $\lambda^k = 1$ and $A^* s = \lambda s$ then ~~$f(x) =$~~

$$f(x) = e^{2\pi i \langle s,x \rangle} + e^{2\pi i \langle A^* s, x \rangle} + \dots + e^{2\pi i \langle (A^*)^{k-1} s, x \rangle}$$

⌂ Check that for $n = 2, 3$ T is ergodic $\Leftrightarrow$ spectrum of $A \cap S^1 = \emptyset$ and T- ergodic $\Leftrightarrow$ spectrum $\subset \mathbb{R}$

3.06.2011

Example of hyperbolic algebraic toral automorphisms

$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad T(x) = Ax \quad T : \mathbb{T}^2 \circlearrowleft$, $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$

$\det A = 1$

$(2-\lambda)(1-\lambda)-1=0$

$\lambda^2 - 3\lambda + 1 = 0$

$\Delta = 5$

$\lambda_1 = \frac{3-\sqrt{5}}{2} < 1$

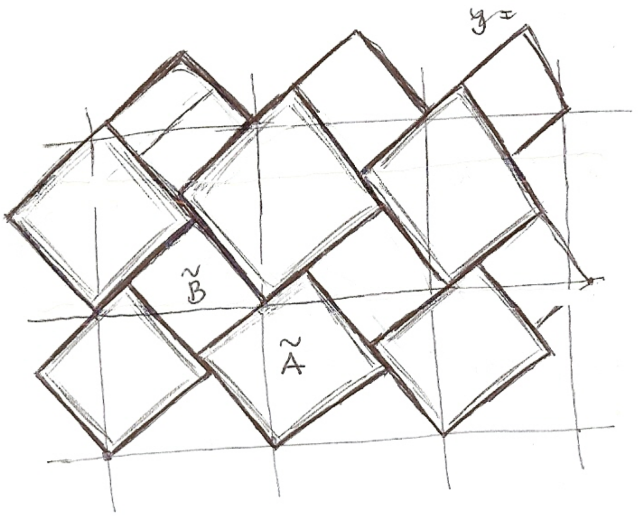
$\lambda_2 = \frac{-3+\sqrt{5}}{2} > 1$

T hyperbolic, leb measure T-invariant, ergodic

$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda_i x \\ \lambda_i y \end{pmatrix} \quad i = 1, 2$

$2x+y = \frac{3-\sqrt{5}}{2} x, \quad 2x+y = \frac{3+\sqrt{5}}{2} x$

$y = \left(\frac{3-\sqrt{5}}{2} - 2\right)x = \frac{-1-\sqrt{5}}{2} x, \quad y = \left(\frac{3+\sqrt{5}}{2} - 2\right)x = \frac{\sqrt{5}-1}{2} x.$



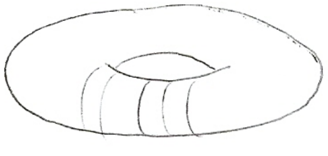
Def: The family of all component of

$T(\mathring{A}) \cap \mathring{A} \qquad T(\mathring{A}) \cap \mathring{B}$

$T(\mathring{B}) \cap \mathring{A} \qquad T(\mathring{B}) \cap \mathring{B}$

($\mathring{A}$ - interior of A)

ther Markov partition for T

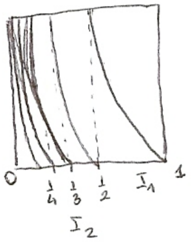


$A = \mathrm{proj}_{\mathbb{T}^2} \tilde{A}$

$B = \mathrm{proj}_{\mathbb{T}^2} \tilde{B}$

$A \cup B = \mathbb{T}^2$

Example: The Gauß map $T(x) = \frac{1}{x} - [\frac{1}{x}]$ $\quad T : (0,1] \to (0,1]$



we checked that the probability measure μ with density $\frac{1}{\ln 2} \frac{1}{1+x}$ is T-invariant

Let $I_n = (\frac{1}{n+1}, \frac{1}{n}]$

$n = 1, 2, \ldots$

$T(x) = \frac{1}{x} - n$ for $x \in I_n$

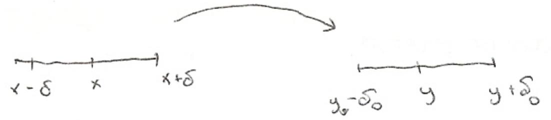
$|T'(x)| = \frac{1}{x^2} \geq n^2$

$\boxed{\text{Fact: } \mu \text{ is ergodic}}$

General idea: Take $A \subset J$ s.t. $T^{-1}(A) = A$, $\mu(A) \neq 0, 1$

$\mu(A) > 0$ so $\mathrm{leb}(A) > 0$. Take $x \in A$ a density point

$y \in J \setminus A$ - " -



we know that T^n - expands.

if we know that $\frac{Leb(I\setminus A \cap (y-\delta_0, y+\delta_0))}{2\delta_0}$ is preserved

with respect to constant - contradiction ξ

if we know $\exists\, c>0$ that $\underbrace{\frac{|(T_j^{-n})'(u)|}{|(T_j^{-n})(v)|} < C}_{\text{bounded distration}}$ $\forall\ u, v \in (y-\delta_0, y+\delta)$ $\forall\ n$ $\forall$ branch T_j^{-n} of T^{-n}

then it is ok.

$$\boxed{\frac{1}{C} < \frac{|(I\setminus A)\cap(y-\delta_0, y+\delta_0)|}{2\delta_0} \qquad \frac{Leb(I\setminus A \cap T^{-n}(y-\delta_0, y+\delta_0))}{|T^{-n}(y-\delta_0, y+\delta_0))} < C}$$

$\tilde{u} = T_j^{-n}(u)$

$\tilde{v} = T_j^{-n}(v)$

we estimate $\frac{|(T^n)'(\tilde{u})|}{|(T^n)'(\tilde{v})|} < ?$, $\underline{|T'| \geqslant 1}$

$$\left|\log\left|\frac{(T^n)'(\tilde{u})}{(T^n)'(\tilde{v})}\right|\right| = \left|\log|(T^n)'(\tilde{u})| - \log|(T^n)'(\tilde{v})|\right| = \left|\log\prod_{i=1}^{n-1}|T'(T^i(\tilde{u}))| -\right.$$

$$\left.-\log\prod_{i=1}^{n-1}|T'(T^i(\tilde{v}))|\right| \leq \sum_{i=1}^{n-1}\left|\log|T'(T^i(\tilde{u}))| - \log|T'(T^i(\tilde{v}))|\right| \leq$$

$$\leq L\sum_{i=1}^{n-1}|T^i(\tilde{u}) - T^i(\tilde{v})| \leq L\sum_{i=1}^{n-1}\frac{1}{Q^{n-i}} < \text{const}$$ independent of n.

suppose: $|T'| \geq Q > 1$ then $|(T_j^{-k})'| \leq \frac{1}{Q^k}$ then

$|u - v| \leq 1 \Rightarrow |T^i(\tilde{u}) - T^i(\tilde{v})| \leq \frac{1}{Q^{n-i}}$

T - the Gauß map

$\mu(A) = \frac{1}{m2}\int_A \frac{1}{1+x}\,dx$ μ - T- invariant, ergodic

$x \in [0,1]$ then $x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3} + \dots}} = [a_1, a_2, \dots]$

$T(x) = \frac{1}{x} - [\frac{1}{x}]$ $a_i = a_i(x) \in \mathbb{N}$

$T([a_1, a_2, \dots]) = [a_2, a_3, \dots]$

$f(x) = \ln \underset{[\frac{1}{x}]}{\overset{\shortparallel}{a_1(x)}} = \ln n$ if $x \in I_n = (\frac{1}{n+1}, \frac{1}{n}]$

$\int f\,d\mu = \sum \ln n\,\mu((\frac{1}{n+1}, \frac{1}{n})) \leq c \sum \ln n\,(\frac{1}{n} - \frac{1}{n+1})$

$\sim \sum \frac{\ln n}{n^2} < \infty$.

so $f \in L^1(\mu)$

By ergodic theorem $S_n = \frac{f(x) + f(Tx) + \dots + f(T^{n-1}x)}{n} \to \int f\,d\mu$

$$S_n = \frac{\ln a_1(x) + \dots + \ln a_1(T^{n-1}x)}{n} = \ln \sqrt[n]{a_1(x) \dots a_1(T^{n-1}(x))} =$$

$$= \ln \sqrt[n]{a_1(x) \dots a_n(x)} \longrightarrow \int f \, d\mu$$

$a_1(T^k x) = a_k(x)$

Corollary: For a.e. $x \in (0,1]$ $\quad \sqrt[n]{a_1(x) \dots a_n(x)} \to c_0 > 0$

$$\frac{a_1(x) + \dots + a_n(x)}{n} \to +\infty$$

$f(x) = a_1(x) = n \quad$ if $\quad x \in I_n = (\frac{1}{n+1}, \frac{1}{n}]$

$f_j = \min(f, j)$

$\int f \, d\mu = \sum \mu((\frac{1}{n+1}, \frac{1}{n}))$ so $f \in L^1(\mu)$

$\sim \sum \frac{1}{n} = \infty$

By Ergodic theorem

$$S_n = \frac{f(x) + f(Tx) + \dots + f(T^{n-1}x)}{n} \to \infty$$