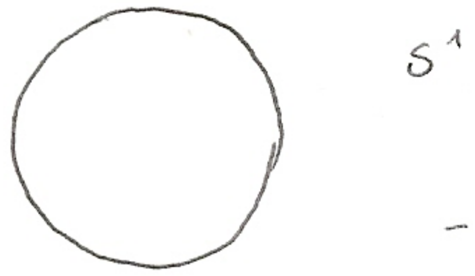
UKŁADY DYNAMICZNE - WYKŁAD 18.02.2011

Dynamical systems
Holomorphic Dynamics - iteration of holomorphic maps in $\hat{\mathbb{C}}$
Geometric theory of measure
Harmonic measure in the plane
Dynamics in $\mathbb{C}\mathbb{P}^k$

Literature:

- „Wstęp do gładkich układów dynamicznych" W. Szlenk
- „I course in dynamical systems", „modern theory of D.S" Katok, Hasselblatt
- Clark Robinson
- F. Przytycki, Urbański (Cambridge)
- Ricardo Mañe „Ergodic theory of D.S"
- Polliott, Yuri (Cambridge)

First example

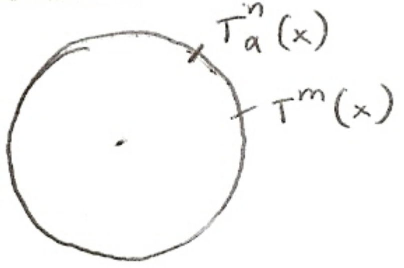


- T_α - rotation by angle α

Proposition: (1) If $\frac{\alpha}{\pi}$ is rational then every trajectory is finite (periodic)

(2) If $\frac{\alpha}{\pi}$ is irrational then every trajectory is dense in S^1

Explanation:



Given $\delta = \frac{1}{N}$, there are $n, m \le N+1$ such that $d(T_\alpha^n x, T_\alpha^m x) < \delta$; (length S^1).

Proposition: For every continuous function $f: S \longrightarrow \mathbb{R}$, then

$$(*) \quad \frac{f + f\circ T_\alpha + \ldots + f\circ T_\alpha^{n-1}}{n} \rightrightarrows \int_{S^1} f\, dl$$

l - normalized Lebesgue measure on S^1

, where T_α is an irrational rotation.

Proof: $f, f\circ T_\alpha, \ldots, f\circ T_\alpha^k, \ldots$ sequence of random variables, equally distributed

$$\mathbb{P}(f\circ T_\alpha < a) = \mathbb{P}(T_\alpha^{-1}(f<a)) = \mathbb{P}(f<a)$$

(1) Consider f: (trigonometric polynomial) $(**)$ $f(z) = z^k$, check that $(*)$ holds for f.

Use density of linear combination of functions $(**)$ in $C(S^1)$.

Corollary: consider the following sequence of measures, $x \in S^1$

$$\nu_n = \frac{\delta_x + \delta_{T_\alpha x} + \ldots + \delta_{T_\alpha^{n-1} x}}{n} \xrightarrow{*} \text{Leb}(S^1) \text{ normalized}$$

$$\nu_n(f) \longrightarrow \text{Leb}(f)$$

~~$\nu_n(f) \Longrightarrow Leb(f)$~~ Def: $T: X \longrightarrow X$ continuous, X- compact, metric

T is called topologically transitive if $\forall\, U, V$ open there exists $n \in \mathbb{N}$ so that $T^n(U) \cap V \neq \emptyset$

Proposition: X- compact, metric, separable space
$T: X \circlearrowleft$ continuous map. Then T is topologically transitive $\Longleftrightarrow$ there exists a dense orbit.

Proof: „$\Leftarrow$" obvious

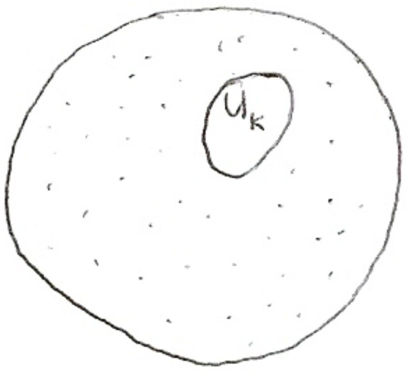
„$\Rightarrow$" since X is separable there exists a countable basis of topology. $\rightarrow$ countable family of open sets U_k such that $\forall x \in X \ \forall\, U \ni x$ there exists $x \in U_k \subset U$.

$R_k = \{ x \in X : \forall n \in \mathbb{N} \ T^n(x) \notin U_k \}$

R_k - closed $\bigcap_n T^{-n}(X \setminus U_k)$

with empty interior

$\operatorname{int} R_k = \emptyset$



From Baire's theorem we get $\bigcup R_k \neq X$

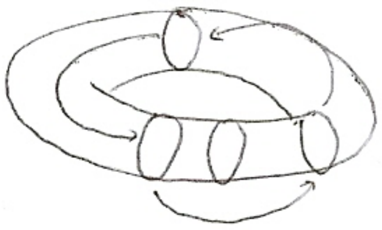
Generalization to $\mathbb{T}^k$:

$\mathbb{T}^k = S^1 \times \ldots \times S^1$

$(z_1 \ldots z_k) \xrightarrow{T} (e^{2\pi i a_1} z_1, e^{2\pi i a_2} z_2, \ldots, e^{2\pi i a_k} z_k)$

Example:

$\mathbb{T}^2$ a_1 rational, $a_1 = 0$
a_2 irrational
$\Big\}$ no dense trajectory
no periodic trajectory



a_1 a_2 both rational all trajectors are periodic

Proposition: (1) if $(a_1, \ldots, a_k)$ are independent over $\mathbb{Z}$

i.e. $(*)$ $n_1 a_1 + n_2 a_2 + \ldots + n_k a_k \in \emptyset\mathbb{Z}$

the only solution of $(*)$ in $\mathbb{Z}$ is $n_1 = n_2 = \ldots = n_k = 0$,
then every trajectory is dense.

(2) if $a_1, \ldots, a_k$ are not independent, then there is no dense trajectory

Proof: $k=2$ a_1, a_2

$n_1 a_1 + n_2 a_2 \in \mathbb{Z}$

then we take the function $\varphi(z_1, z_2) = z_1^{n_1} \pm z_2^{n_2}$

$\varphi \circ T(z_1, z_2) = z_1^{n_1} \cdot z_2^{n_2} e^{2\pi i (n_1 a_1 + n_2 a_2)} = z_1^{n_1} \cdot z_2^{n_2}$. φ is T- invariant

$\Rightarrow$ T is not transitive

25.02.2011

$f: S^1 \to S^1$

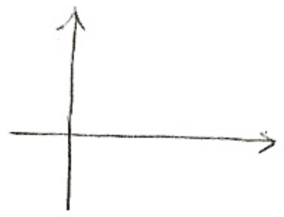
continuous, $\deg f = 1$

f monotone

$F(x+1) = F(x) + d$

$F: \mathbb{R} \to \mathbb{R}$

$F(\mathbb{Z}) \subseteq \mathbb{Z}$



$S^1 = \mathbb{R}/\mathbb{Z}$

Def: $f: M \to M$ orientable manifold, $x \in M$ regular value

$\forall\, f(x) = x$ Df invertible, $\deg = \sum_{x \in f^{-1}(x_1)} \operatorname{sign} f'(x)$

$G: \mathbb{R} \to S^1$

$\theta \to e^{2\pi i \theta}$

$\mathbb{R}/\mathbb{Z} \leftrightarrow S^1$

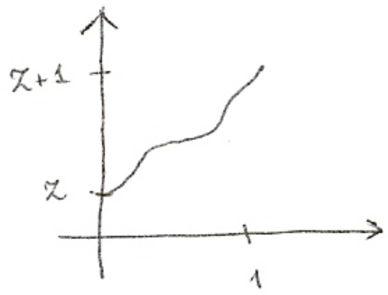
Fact: $\exists\, F: \mathbb{R} \to \mathbb{R}$ $\forall x$ $F(x+1) = F(x) + 1$

$$\begin{tikzcd} {} \arrow[r, "F"] \arrow[d, "G"'] & {} \arrow[d, "G"] \\ {} \arrow[r, "f"'] & {} \end{tikzcd}$$

lift

commites

$\tilde{F} = F + m$ integer



H. Poincaré

$\rho(F) = \lim_{n \to \infty} \frac{F^n(x)}{n}$ - rotation number (liczba obrotu)

Theorem 1: $\rho(F)$ exists, does not depend on x

Example: $f_\alpha(x) = x + \alpha \pmod 1$

$f_\alpha(z) = z \cdot e^{2\pi i \alpha}$, $|z| = 1$

$\rho(f_\alpha) = \alpha$

Simple properties: $F(x+m) = F(x) + m$

$F + k$: $\rho(F+k) = \rho(F) + k$

$\cdot F$ monotone, increasing

Proof of thm 1:

$\forall q$ $\exists p$ $q, p \geq 0$ integers

$p \leq F^q(0) \leq p+1$

$F^{2q} \leq F^q(0) + p + 1 \leq 2(p+1)$

$\vdots$

$F^{nq}(0) \leq n(p+1)$, $F^{nq}(0) \geq np$

$\frac{p}{q} \leq \frac{1}{nq} F^{nq}(0) \leq \frac{p+1}{q}$

$\frac{p}{q} \leq \liminf_{n \to \infty} \frac{F^{nq}(0)}{nq} \leq \limsup_{n \to \infty} \ldots \leq \frac{p+1}{q}$ $nq + r$, $0 \leq r < q$

ka $\frac{p}{q} \le \liminf_{k} \frac{F^k(0)}{k} \le \limsup_{k} \frac{F^k(0)}{k} \le \frac{p+1}{q}$

q - arbitrary so $\forall q$ $(\limsup - \liminf) \le \frac{1}{q}$

so lim exists. Does not ~~exis~~ depend on x, $0 \le x \le y \le 1$

$$\underset{\substack{\| \\ \rho(F)}}{\rho(F,0)} \le \rho(F,x) \le \rho(F,y) \le \underset{\substack{\| \\ \rho(F)}}{\rho(F,1)}$$

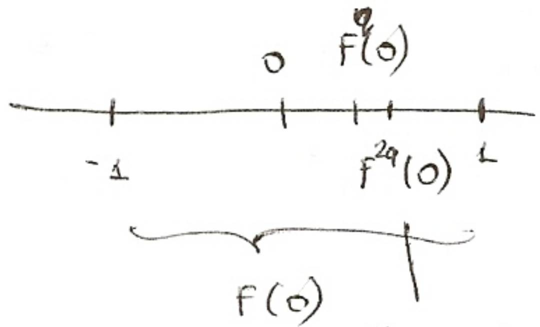
Theorem 2:

$\rho(f) \in \mathbb{Q} \iff \exists$ a periodic point for f

$\exists q, x$ $f^q(x) = x$, $F^q(x) = x + p$

Proof:

„$\Rightarrow$" for simplicity let's assume that $\rho(F) = 0$, $q\,\rho(F) = p/q$ - ex.



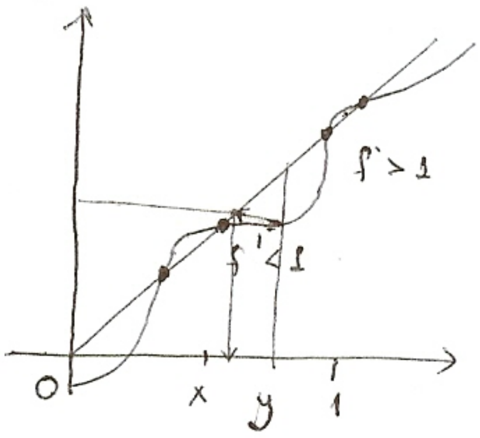
it won't be smaller then 1

$$F^{(n+1)q}(0) - F^{nq}(0) \to 0$$
$$\downarrow$$
$$F^q(x) = x$$

then $\exists$ a fixed point for F

$F^q(x) - p = x$

$\rho(F) = p/q$



Theorem(3): There exists $\exists h : S^1 \to S^1$, continuous, monotone, $\deg h = 1$

$R_\alpha(x) = x + \alpha$ (mod 1) $\forall f$ - monotone, $f: S^1 \to S^1$, deg 1, continuous, $\rho(f) \notin \mathbb{Q}$

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "h"'] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "R_{\rho(f)}"'] & \cdot \end{tikzcd}$$

h is called semiconjugacy

Theorem 4: If f is diffeomorphism, $\operatorname{var} \log|f'| < \infty$, then h is homeo (h from thm 3).

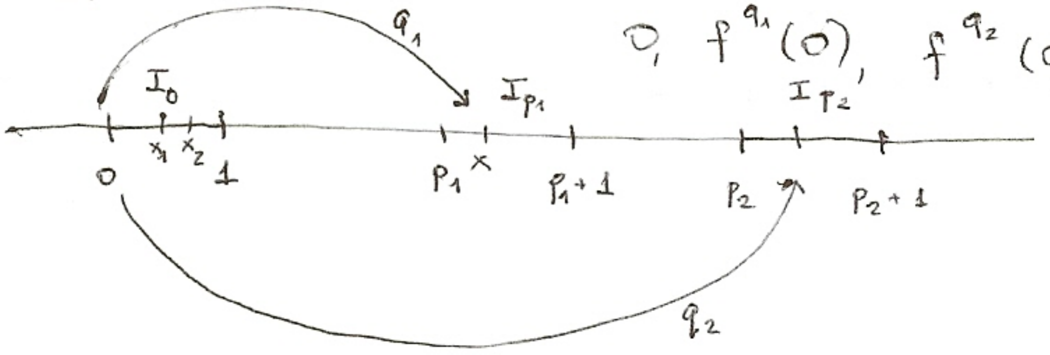
Remark: $C^2 \Rightarrow f'$ Lipschitz

How smooth h is

How smooth h is

$|\rho(f) - p/q| \geq \frac{c}{q^m}$ diophantine condition

Proof of thm 3:



$0, f^{q_1}(0), f^{q_2}(0)$

deg is preserved

$F^{q_2-q_1}(x) \geq p_2 - p_1$

define h on any trajectory $0, f(0), \ldots \rightarrow 0, \rho(f), 2\rho(f), 3\rho(f), \ldots$

monotone

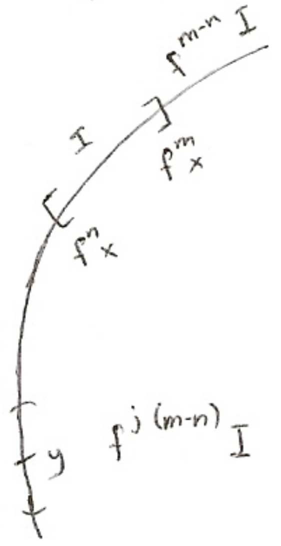
4.03.2011

Twierdzenie (Denjoy): Jezeli $f: S^1 \circlearrowleft$ jest zachowującym orientację dyfeomorfizmem klasy C^2 (a nawet wystarczy mniej tzn ze $\log|f'|$ ma ograniczone wahanie) z niewymierną liczbą obrotu ρ, to f jest topologicznie sprzężone z obrotem R_ρ

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "h"'] & S^1 \arrow[d, "h"] \\ S^1 \arrow[r, "R_\rho"] & S^1 \end{tikzcd}$$

Lemacik: Niech f będzie zachowującym orientację homeomorfizmem okręgu z $\rho(f) \notin \mathbb{Q}$, $x \in S^1$, $I = [f^n x, f^m x]$ dla pewnych $n, m \in \mathbb{Z}$

wówczas $\forall\, y \in S^1 \ \exists\, k \in \mathbb{Z}$ t.ze $f^k y \in I$.



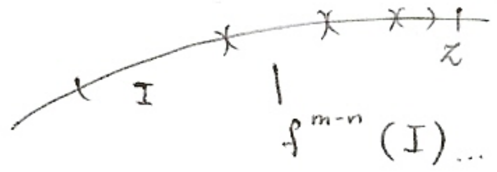
D:

Rozpatrzmy $I, f^{m-n}(I), f^{2(m-n)} I, \ldots$

albo $\bigcup_{j \in \mathbb{Z}} f^{j(m-n)} I = S^1$

wtedy dobrze

albo $\bigcup_{..} \neq S^1$



wówczas $z = \lim_j f^{j(m-n)} \underbrace{(f^n x)}_{v} \Rightarrow f^{(m-n)}(z) = z$

$\tilde{f} := f^{(m-n)}$

$\lim_j \tilde{f}^j v = z$

sprzeczność z niewymiernością $\rho(f)$.

Stwierdzenie: f - zachowujący orientację homeomorfizm okręgu, $\rho(f) \notin \mathbb{Q}$, to

(a) $\omega(x)$ nie zależy od x

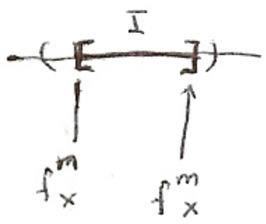
(b) $\omega(x) = S^1$ albo $\omega(x)$ jest doskonałym, nigdziegęstym, domkniętym podzbiorem S^1.

(c) (ogólne prawda) $\omega(x)$ jest f- niezmienniczy

$$f(\omega(x)) = f^{-1}(\omega(x)) = \omega(x)$$

<u>Dowód:</u> (c) jasne

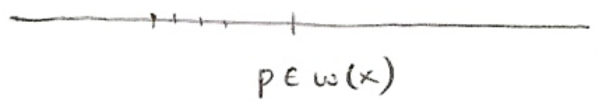
(b) jeżeli $\operatorname{int}(\omega(x)) \neq \emptyset$



powtarząc argument z poprzedniego lemacika dostajemy, że w tym przypadku $\omega(x) = S^1$.

Doskonałość tj brak punktów izolowanych

$\omega(x) = \omega(p) \ni f^{n_i}(p) \longrightarrow p$



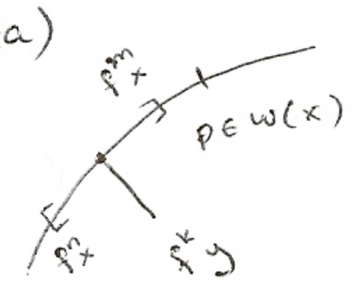
Jak znaleźć ciąg punktów $q_i \in \omega(x)$

$q_i \longrightarrow p$

(a) => $\omega(x) = \omega(p)$

$f^{n_i}(p) \rightarrow p$ z niezmienniczości $f^{n_i}(p) \in \omega(p) = \omega(x)$. wskazaliśmy ciąg punktów zb. do p.

(a)



weźmy dowolne $y \in S^1$. Pokazujemy, że $p \in \omega(y)$

wówczas istnieje $k \in \mathbb{N}$ t.że (z lemacika)

$f^k(y) \in [f^n_x, f^m_x]$

zatem istnieje ciąg $f^{k_i} y \longrightarrow p$.

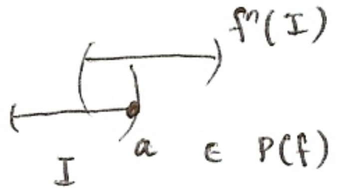
<u>Stwierdzenie:</u> Jeżeli I jest składową ~~uzu~~dopełnienia $J(f) = S^1 \setminus P(f)$ to I jest odcinkiem błądzącym (wandering)

tzn $f^n(I) \cap I = \emptyset \quad \forall n \in \mathbb{Z}$

$$\Downarrow$$

$$f^n(I) \cap f^m(I) = \emptyset \quad n \neq m$$

<u>Dowód:</u>



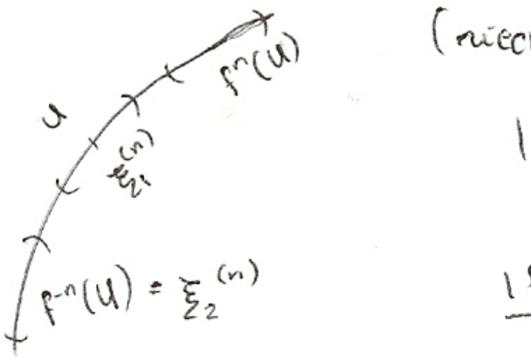
a: Koniec I $a \in P(f)$

Jeżeli $f^n(I) \cap I \neq \emptyset$ to $a \in f^n(I)$, ale przecież $f^n(I)$ jest rozłączne z $P(f)$.

<u>Stwierdzenie:</u> (główny krok w dowodzie Denjoy)

Jeżeli f spełnia założenia tw. Denjoy, to $P(f) = S^1$

<u>Dowód:</u> Jeżeli $P(f) \neq S^1$ to istnieje odcinek $U \subset S^1 \setminus P(f)$ (niech U będzie składową $S^1 \setminus P(f)$)

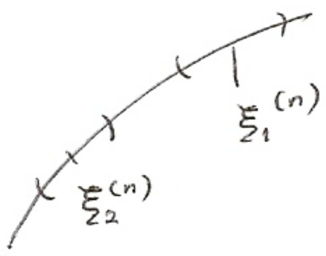


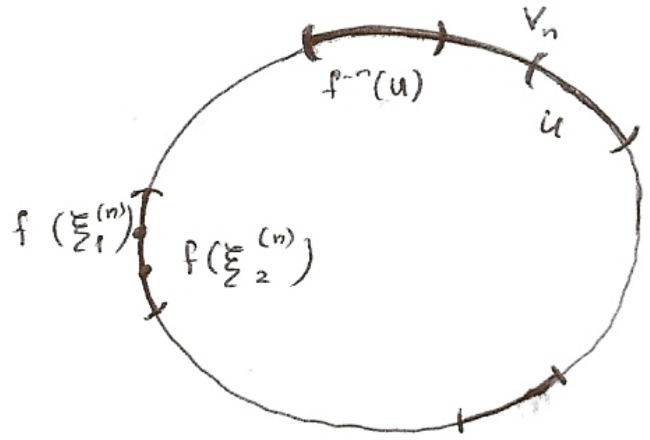
$$\frac{|f^n(U)|}{|U|} = |(f^n)'|(\xi_1^{(n)}) \quad \text{dla pewnego } \xi_1^{(n)} \in U$$

$$\frac{|f^{-n}(U)|}{|U|} = \frac{1}{|(f^n)'|(\xi_2^{(n)})} \quad \text{dla pewnego } \xi_2^{(n)} \in f^{-n}(U)$$

$$\left|\log \frac{|f^n(U)|\cdot|f^{-n}(U)|}{|U|^2}\right| \; (**) = \left|\log \frac{|(f^n)'(\xi_1^{(n)})|}{|(f^n)'(\xi_2^{(n)})|}\right| = \left|\log \frac{\prod_j (f')(f^j(\xi_1^{(n)}))}{\prod_j (f')(f^j(\xi_2^{(n)}))}\right| =$$

$$= \left|\sum_{j=0}^{n-1} \left(\log(f')(f^j(\xi_1^{(n)})) - \log(f')(f^j(\xi_2^{(n)}))\right)\right| \; (*)$$



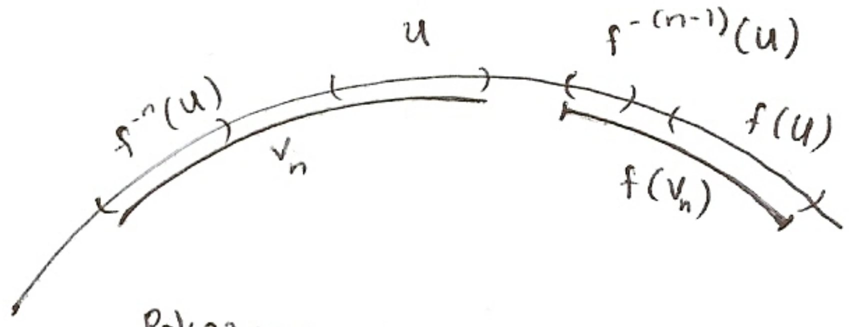


Pokażemy, że istnieje nieskończenie wiele n takich, że łuki będące $\xi_1^{(n)}$, $\xi_2^{(n)}$ mają rozłączne obrazy f^j dla $j = 0, 1, \dots, n-1$ stąd będzie wynikało, że $(*)$ jest ograniczona przez stałą (nieskończone wahanie $\log f'$)

$\xi_1^{(n)} \in U$

$\xi_2^{(n)} \in f^{-n}(U)$

Tymczasem $(**)$ jest nieograniczone przy $n \to \infty$. Stąd sprzeczność.



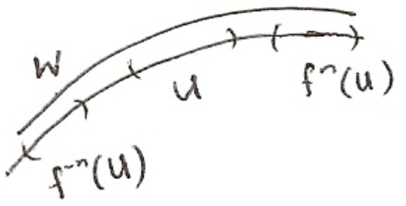
Pokażemy, że V_n, $f(V_n), \dots, f^{(n-1)}(V_n)$ są rozłączne (jeśli n wybierzemy odpowiednio)

~~Pokażemy~~, zauważmy, że $f^{-(n-1)}(U)$ nie może wpaść do V_n

Wówczas będziemy mieli $f(V_n) \cap V_n = \emptyset$

$f^2(V_n)$ musimy zadbać, żeby było rozłączne z V_n, $f(V_n)$

warunek wystarczający na to, żeby odcinki $\underline{V_n, f(V_n), \dots, f^{n-1}(V_n)}$ $(*)$ były parami rozłączne

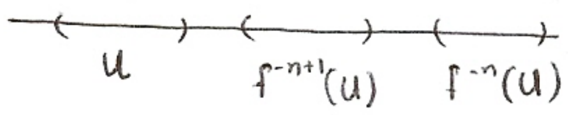


w tym łuku nie może być wcześniejszych obrazów $f^j(V_n)$ dla $|j| < n$.

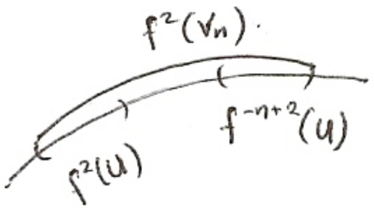
11.03.2011

End of the proof of Denjoy theorem

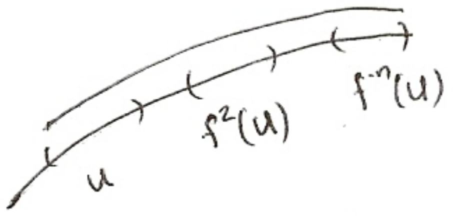
If $f(V_n) \cap V_n \neq \emptyset \Rightarrow f^{-n+1}(U)$ falls in V_n



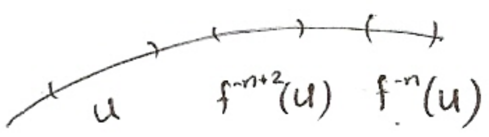
$f^2(V_n) \cap V_n = \emptyset$



If $f^2(V_n) \cap V_n \neq \emptyset$ then either $f^2(U) \cap V_n \neq \emptyset$ or $f^{-n+2}(U) \cap V_n \neq \emptyset$



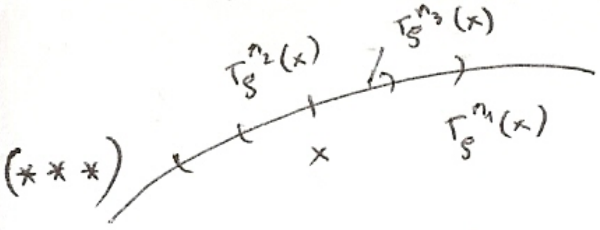
or



In order to verify $(*)$ we need to choose n so that

$$\left.\begin{matrix} f(U), f^2(U), \dots, f^{+n-1}(U) \\ f^{-n+1}(U), f^{-n+2}(U), \dots, f^{-1}(U) \end{matrix}\right\} \text{ do not intersect } V_n \quad (**)$$

Let us concider the rotation T_ϱ , $\varrho = \varrho[f)$



choose n such that $d(T_\varrho^n x, x) < d(x, T^j x)$ for all $1 \leq j < n-1$

$$\Rightarrow d(T_\varrho^{-n} x, x) = d(T_\varrho^n x, x) < d(x, T_\varrho^j x) = d(x, T_\varrho^{-j} x)$$

If n is chosen like this then the $(*)$ is satisfied since $(* * *)$ implies that $(* *)$ is satisfied.

Why $(* * *) \Rightarrow (* *)$? we have a semiconjugacy

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "h"] & S^1 \arrow[d, "h"] \\ S^1 \arrow[r, "T_\varrho"] & S^1 \end{tikzcd}$$

Choose an arbitrary point $z \in \mathcal{U}$, denote $x \in h(z)$ the order of points $z, f^{\pm 1}(z), f^{\pm 2}(z), \ldots$ in S^1 is the same as the order of points $T^j(x)$

∎

$\varrho(f) \in \mathbb{Q}$

Def: $f: S^1 \circlearrowleft$ C^1- diffeomorphism preserving orientation is called Morse - Smale diffeomorphism if

(1) $\varrho(f) \in \mathbb{Q}$

(2) all periodic points are non - degenerated (if $f^q(p) = p$ then $|f^q(p)|' \neq 1$)

(equivalently if F is a lift and $F^q(x) = x + p$, $(F^q)'(x) \neq 1$

$$\begin{tikzcd} x \arrow[d] & \mathbb{R} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] \\ x & S^1 \arrow[r, "f"] & S^1 \end{tikzcd}$$

Def: Let f be a C^1- diffeomorphism preserving orientation. f is C^1 - structurally stable if there is a C^1- neighbourhood $\mathcal{U}$ of f such that for every $\tilde{f} \in \mathcal{U}$ is topologically conjugate to f.

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "h"'] & S^1 \arrow[d, "h"] \\ S^1 \arrow[r, "\tilde{f}"] & S^1 \end{tikzcd}$$

h- homeo

Excersize: verify that diff are open in C^1- topology

Theorem: C^1- diffeomorphism $f: S^1 \circlearrowleft$ (preserving orientation) is C^1- structurally stable if f is M-S diffeomorphism. Moreover : C^2 - M-S are C^2- dense in the space of C^2 oriental preserving diffeomorphisms $S^1 \circlearrowleft$.

Corollary: since every C^1 -diffeomorphism $f: S^1 \circlearrowleft$ can be C^1 - approximated by C^2 - diffeomorphisms , this implies that M-S diff are C^1 - dense in the space $\mathrm{diff}^1(S^1)$.

Proof: First we prove that if f is M-S C^1- diffeomorphism then f is C^1- structurally stable.

Let $\rho(f) = \frac{p}{q}$ First we replace f by $g = f^q$, then $\rho(g) = 0$ and there is a lift G of g such that the only points X satisfying $G^m X = X + m$ (for some m, n) are fixed points $x_1, \dots, x_N = x_1 + 1$.

~~Given~~ $\exists\, \varepsilon > 0$ s.t. one can find neighbourhoods $U_i \ni x_i$ so that

(1) $|G(x) - x| > \varepsilon$ $x \notin \bigcup_i U_i$

(2) $|G'(x) - 1| > \varepsilon$ in U_i's

There, ~~if~~ exists δ so that if $\tilde{f}$ is δ - C^1 close to f then there exists a lift $\tilde{F}$ of $\tilde{f}$ which is C^1- close to F and a lift $\tilde{G}$ of $\tilde{f}^q$ ($\tilde{G} = \tilde{F}^q + p$) which is $\frac{\varepsilon}{2}$ C^1- close to G

for some integer

then observe that

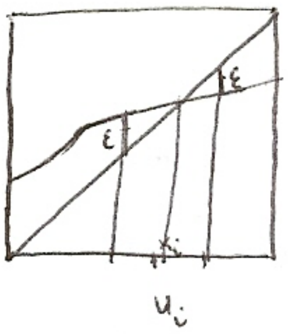
(1) $|\tilde{G}'(x) - 1| > \frac{\varepsilon}{2}$ on $\bigcup U_i$

(2) At the end points of U_i's the graph of $\tilde{G}$ is on the same side of the diagonal as G

(3) $|\tilde{G}(x) - x| > \frac{\varepsilon}{2}$ outside $\bigcup_i U_i$

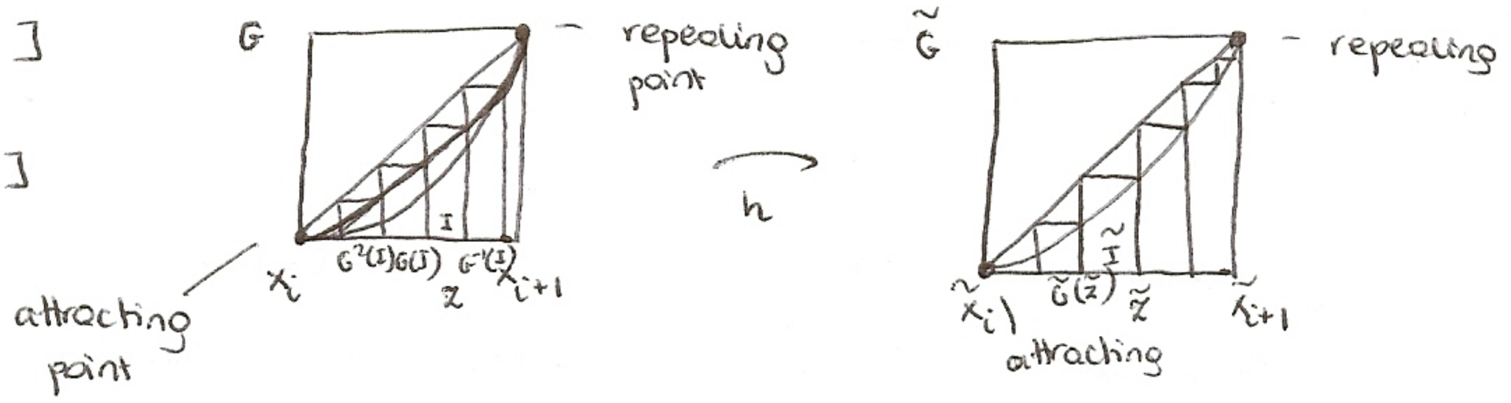
(1) → (3) implies that $\tilde{G}$ has exactly one fixed point in each U_i

$\tilde{G}(x) - x = 0$
$\tilde{G}'(x) - 1 \neq 0$ in U_i



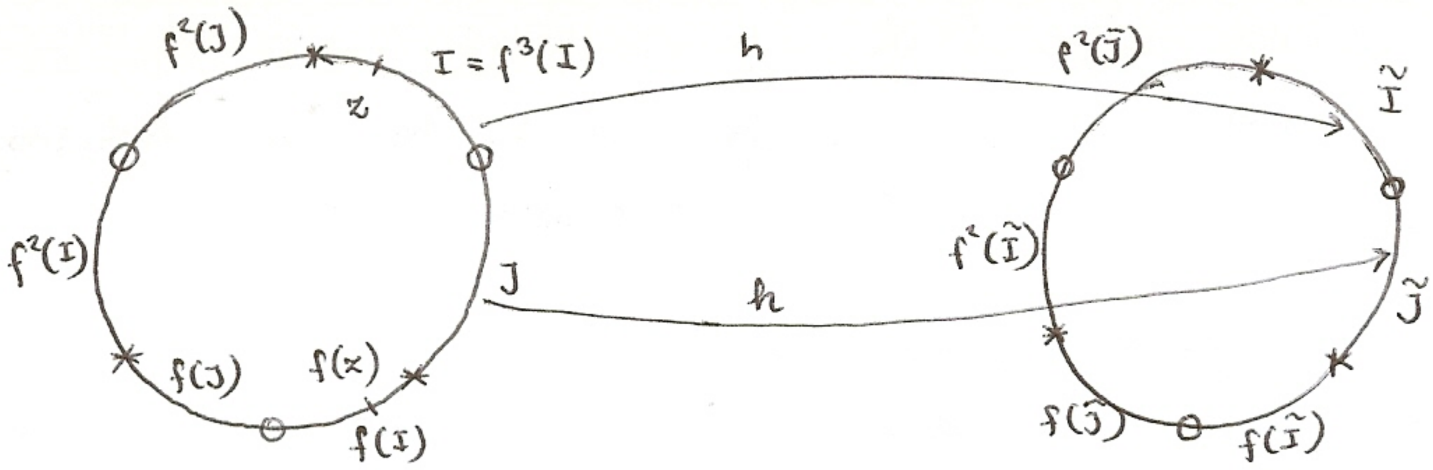
Let $\tilde{x}_1, \dots, \tilde{x}_N$ be fixed points of $\tilde{G}$

How to construct the conjugacy between G and $\tilde{G}$ on the intervals $[x_i, x_{i+1}]$
↓
$[\tilde{x}_i, \tilde{x}_{i+1}]$



$h_i : I \to \tilde{I}$ an arbitrary homeomorphism (preserving orientation)

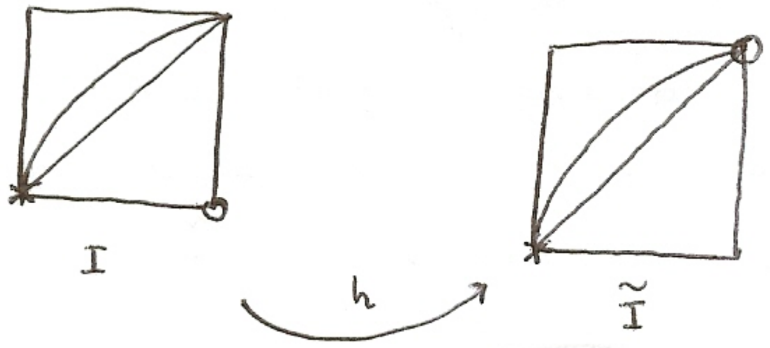
$h(G^j(w)) = \tilde{G}^j(h(w))$



f has 2 periodic orbits of period (3)

- $*$ repelling
- o attracting

Graph of $f^3|_I$ (or rather it's lift G)



we construct the conjugacy $h : I \to \tilde{I}$ as before

How to define h on $f(I)$? we have to put $h(f(x)) = \tilde{f}(h(x))$, $x \in I$

Similarly we define h on $f^2(I)$ putting $h(f^2(x)) = \tilde{f}^2(h(x))$

Final observation h is a conjugacy between f and $\tilde{f}$

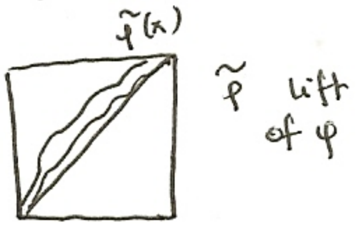
Similarly on $f(J)$, $f^2(J)$

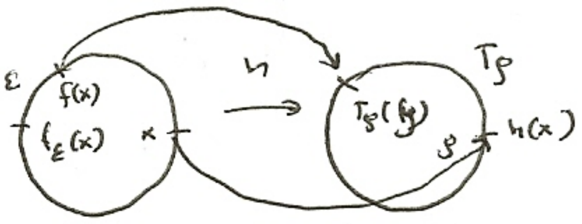
This is a conjugacy between f and $\tilde{f}$ because we have to put $h(f^3(x)) = \tilde{f}^3(h(x))^{(*)}$ at the points $f^3(x)$ $(x \in I)$

But h was already defined on I, since h was a conjugacy between f^3 and $\tilde{f}^3$ on I, this means that $(*)$ is satisfied for h which we started with

18.03.2011 „=>"

step 2 given f- C^2 diffeo with $\rho(f)$ irrational. How to cheack that f is not (C') stable ?





$f_\varepsilon = T_\varepsilon \circ f$

by denjoy thm: $h \circ f_\varepsilon \circ h^{-1} = \underbrace{\varphi \circ T_e}_{\text{conjugate of } f_\varepsilon}$

Conclusion: $\varphi(f_\varepsilon) \neq \rho(f)$

step 3: Therefore f_ε is not topologically conjugate to f. f-arbitrary C' diffeo with $\rho(f) \notin \mathbb{R}$? Assume that f is C'- structurally stable. Then $\exists u$, $\exists f$ in C'-topology so that every $g \in V$ $g \approx f$, in particular $\rho(g) = \rho(f)$. But C^2- diffeo are dense in C'-topology. One can choose $g \in V$ s.t. $g \in C^2$

See previous figure

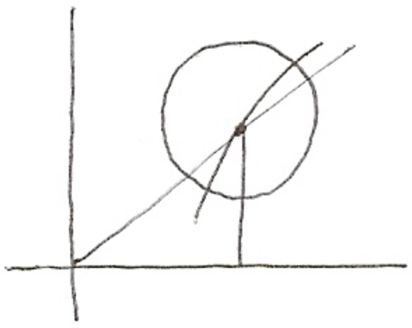
g can be perturbed to $g_\varepsilon \approx g$ so that $\rho(g) \neq \rho(g_\varepsilon)$

<u>Step 4</u> density of M-S diff in C^1- topology. Let f be an arbitrary C^1 diff. Let $\mathcal{U} \ni f$ be an arbitrary neighbourhood of f.

First we find C^2 diff $g \in \mathcal{U}$. Either $\rho(q) \in \mathbb{Q}$ or $\rho(q) \notin \mathbb{Q}$
if $\rho(g) \notin \mathbb{Q}$ then again we concider g_ε as before

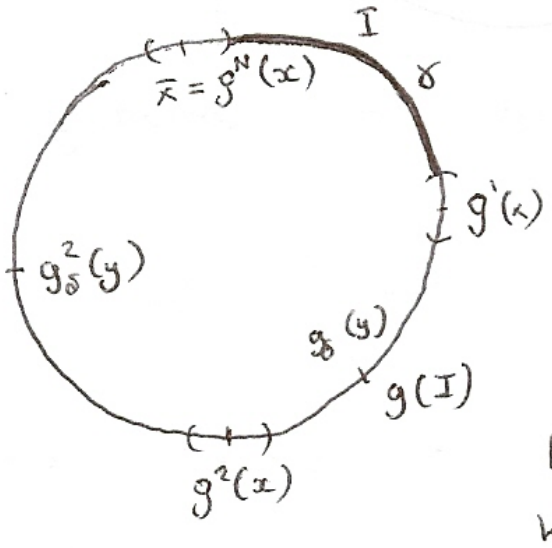
$$\varepsilon \to g_\varepsilon \longrightarrow \underset{\text{rational}}{\rho(g_\varepsilon)} \qquad \rho(g)$$

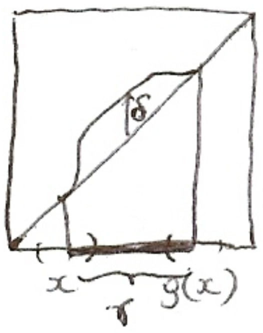
If all periodic orbits of g_ε are non-degenerated then g_ε is M-S,
if not, then we choose one periodic orbit and we perturb g_ε slightly in C^1- topology so that this point x remains remains periodic and becomes non-degenerated



$(g^2)'(x) \neq 1$

Call this perturbed map





Let $k: S^1 \to S^1$ be defined
$k = id$ outside γ
$k =$ as in the picture on γ

$\rho(g) = p/q$

Now, every periodic orbit of g has period q (in the picture, $q = 3$) and it crosses the arc γ, y - period point in (γ)

$$\begin{tikzcd} \mathbb{R} \arrow[r, "G"] \arrow[d] & \mathbb{R} \arrow[d] \\ S^1 \arrow[r, "g"] & S^1 \end{tikzcd}$$

$G^q(Y) = Y + p$ $\qquad Y$ is periodic for G

$G_\delta^q(Y) = Y + p + \delta$

g_δ - is a perturbation of g

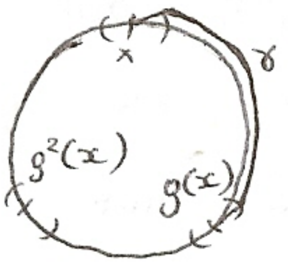
$g_\delta = k_\delta \circ g$

If Y is so that $y = T(Y) \in \gamma$
then (1) y is periodic for g iff
$G^q(Y) = Y + p$
(2) y is periodic for g_δ iff
$G^q(Y) = Y + p + \delta$

we want all periodic ~~points~~ orbits of g_δ to be non-degenerated

$G_\delta^q \neq (Y) = Y + p + \delta$

and $(G_\delta^q)'(Y) \neq 1$



$y \neq x$

Let $L(Y) = G^q(Y) - Y$

we see that $y = \pi(Y)$ is a degenerate periodic point of G_δ iff

$L(Y) = \delta + p$ and $L'(Y) = 0$

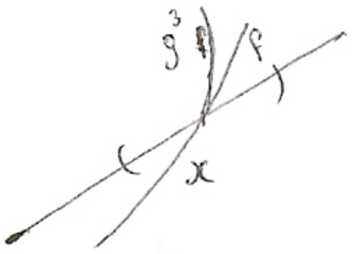
<u>Conclusion:</u> If δ is chosen so that $p + \delta$ is not a critical value of L ~~G^q~~, then the map G_δ has no degenerate periodic orbits, i.e. g_δ is a M-S dyffeomorphism.

we need Sard's lemma

<u>Lemma:</u> If $f: ~~\mathbb{R}~~ [0,1] \longrightarrow \mathbb{R}$ is a C^2 map then

Leb (critical values of f) = 0

$\Rightarrow$ g_δ has no degenerate periodic orbints in γ



x - fixed point of f

$f'(x) \neq 1$

Then there exists a neighbourhood $V \ni x$ and $\delta > 0$ so that if f_δ is a C^1- perturbation of f, f_δ is C^1 close to f then f_δ has only one fixed point in V

To finish the proof we need to know that if f is C^1- str. stable $\Rightarrow$ f is M-S . we know already

(1) If is. str. stable then $\rho(f) \in \mathbb{Q}$

(2) If f is stable then f has finitely many periodic orbits

(3) If one of these periodic points is degenerote then one can perturb f in C^1- topology so that the numbers of periodic points changes

$\Rightarrow$ f cannot be struct - stable

<u>25. 03. 2011</u>

<u>dimension 2</u>

Diff of $\mathbb{T}^2$

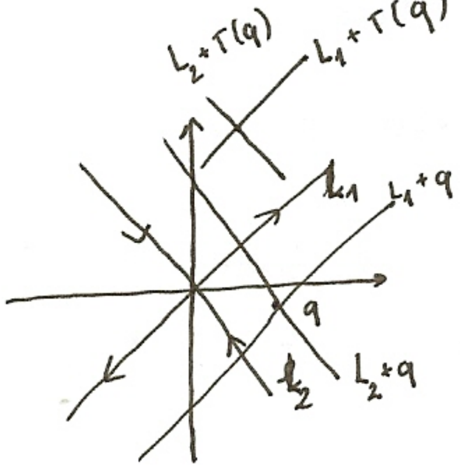
<u>Example:</u> (important)

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ linear map, ex of a linear hyperbolic automorphism of $\mathbb{R}^2$

$T(x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x$

$\lambda_1 = \frac{3+\sqrt{5}}{2} > 1$ $\quad \lambda_2 = \frac{3-\sqrt{5}}{2} < 1$

$y = \frac{\sqrt{5}-1}{2} x$ $\quad y = -\frac{\sqrt{5}+1}{2} x$



$T(\mathbb{Z}^2) = \mathbb{Z}^2$

$\det T = 1$

$T^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "T"] \arrow[d, "\pi"'] & \mathbb{R}^2 \arrow[d, "\pi"] \\ \mathbb{T}^2 \arrow[r, "f"] & \mathbb{T}^2 \end{tikzcd}$$

$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

T induces a diffeomorphism $\mathbb{T}^2 \circlearrowright$

$f = \pi \circ T \circ \pi^{-1}$

$T(L_1 + q) = L_1 + T(q)$

Def. Let $f : M \circlearrowright$ diff C^1, a fixed point p $(f(p) = p)$ is called *hyperbolic* iff $\operatorname{spec}(D_p f : T_p M \circlearrowright)$ does not contain eigenvalues of mod 1.

Ex: Point $(0,0)$ is a hyperbolic fixed point of T

Def: If p - periodic, with period n then is hyperbolic if $D_p f^n : T_p M \circlearrowright$ has no eigenvalues of mod 1

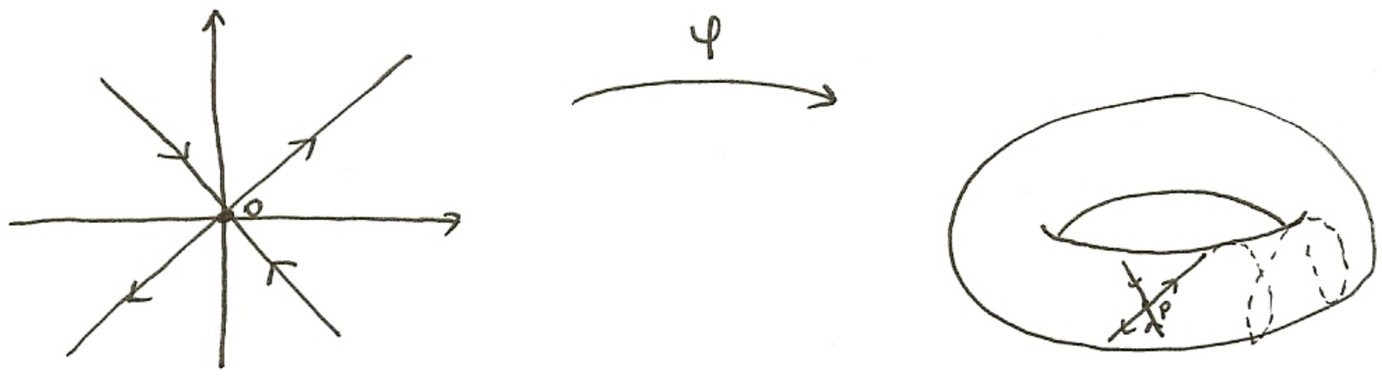
Def: f is a diffeomorphism of a smooth manifold M, $M \subset \mathbb{R}^N$. Let $\Lambda \subset M$ be a closed ~~compact~~ f-invariant subset. We say that Λ is a *hyperbolic set* if for every $p \in \Lambda$ there is a splitting $T_p M = E_p^s \oplus E_p^u$ so that

(1) $D_p f(E_p^s) = E_{f(p)}^s$

(2) $D_p f(E_p^u) = E_{f(p)}^u$

and $\exists\ c > 0, \lambda > 1$ so that $\overbrace{}^{\forall p \in \Lambda}$ $\| D_p f^n(v) \| \leq c \lambda^{-n} \|v\|$, $v \in E_p^s$

$\| D_p f^{-n}(v) \| \leq c \lambda^{-n} \|v\|$, $v \in E_p^u$



$D_0 \varphi(L_1) = E_p^u$

$D_0 \varphi(L_2) = E_p^s$

(not M-S system)

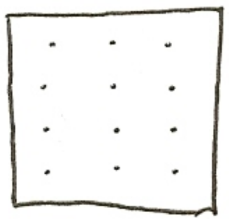
$\varphi(L_1) = W^u(\overset{\times}{p}, f)$ - unstable manifold of p

$\varphi(L_2) = W^s(p, f)$ - stable manifold of p

Proposition: f induced by T (from example) on $\mathbb{T}^2$ $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

f has a dense subset of periodic orbits

Proof:



fundemental square

consider the set

$Qq = \left(\frac{m_1}{n}, \frac{m_2}{n}\right) \subset \mathbb{R}^2$

then $T(Q) = \left(\frac{\tilde{m}_1}{n}, \frac{\tilde{m}_2}{n}\right)$

Thus the trajectory of $q = \pi(Q)$, $q \in \mathbb{T}^2$ is finite

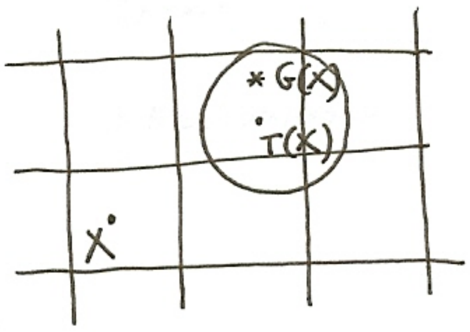
These are all periodic points if $x \in \mathbb{T}^2$ periodic

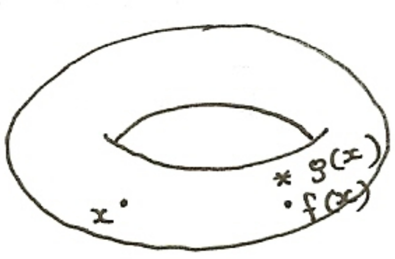
$$\begin{tikzcd} X \arrow[d, "\pi"] \\ x \end{tikzcd} \qquad T^n(X) = X + \underset{\overset{\cap}{\mathbb{Z}^2}}{(m,n)}$$

$$(T^n - I)(X) \in \mathbb{Z}^2 \quad \Rightarrow \quad X = (T^n - I)^{-1}(m,n)$$

<u>Theorem:</u> The diff f (induced by T) is C^1 - structurally stable

<u>Proof:</u> Let $g: \mathbb{T}^2 \circlearrowleft$ be a diffeomorphism which is C^1- ~~closed~~ close to f. We claim that there exists a map $G: \mathbb{R}^2 \circlearrowleft$ which is C^1- close to T such that $\pi \circ G = g \circ \pi$, locally $G = \pi^{-1} \circ g \circ \pi$





G is a small C^1 - perturbation of T

We refer to Grdoman - Harfman theorem (on local stability)

<u>Thm:</u> (1) (version for diff)

Let f be a C^1 diffeomorphism defined in a neighbourhood of 0 (in $\mathbb{R}^n$), $f(0) = 0$, 0 is a hyperbolic fixed point.

Let $L = D_0(f)$

Then $\exists\, U \ni 0$ in $\mathbb{R}^n$ and a homeomorphism H, $H(0) = 0$, $H: U \to H(U)$

and

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "h"'] & U \arrow[d] \\ h(U) \arrow[r, "L"] & L(H(U)) \end{tikzcd} \qquad H \circ f = L \circ H$$

(2) (version for ~~floats~~ flows)

Let X be a C^1 vector field defined in a neighbourhood of 0 (in $\mathbb{R}^n$), $X(0) = 0$, assume that $A = D_0X$ has no eigenvalues λ with $\mathrm{re}\,\lambda = 0$. Then $\exists\, U \ni 0$, and a homeomorphism H on U so that

$$\begin{tikzcd} U \arrow[r, "\varphi_t"] \arrow[d, "H"'] & {} \arrow[d, "H"] \\ {} \arrow[r, "\psi_t"] & {} \end{tikzcd}$$

φ_t - flow on the field X

ψ_t - flow defined by $\dot{x} = Ax$

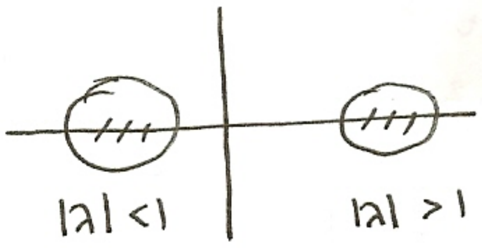
$\psi_t(x) = \exp(tA)(x)$

Outline of proof of G-H thm:

<u>Lemma 1:</u> Let L be hyperbolic, linear automorphism $L: \mathbb{R}^n \circlearrowleft$, then there exists a norm in $\mathbb{R}^n$ so that

$\|L|_{E^s}\| < a < 1$

$\|L^{-1}|_{E^u}\| < a < 1$



$$\mathbb{R}^n = E^s \oplus E^u$$

Lemma 2: Let B be a Banach space. Let $\mathcal{L}: V \to V$ linear operator so that $\|\mathcal{L}\| < a < 1$. Let G be an invertible linear operator so that $\|G^{-1}\| < a < 1$ then

(a) $I + \mathcal{L}$ is invertible and $\|(I+\mathcal{L})^{-1}\| \leq \frac{1}{1-a}$

(b) $I + G$ is invertible and $\|(I+G)^{-1}\| \leq \frac{a}{1-a}$

Proof: excersize

Proposition: Let L be a linear hyperbolic automorphism in $\mathbb{R}^n$. Then $\exists\, \varepsilon > 0$ such that if $\varphi \in C_b(\mathbb{R}^n)$ $(\varphi: \mathbb{R}^n \to \mathbb{R}^n)$ and φ is Lipschitz continues with constant $\leq \varepsilon$ then L and $L + \varphi$ are topologically conjugate in $\mathbb{R}^n$.

Proof of prop:

We search for $H = I + u$, $u \in C_b(\mathbb{R}^n)$ continuous, bounded

$(I+u) \circ L = (L+\varphi)(I+u)$

$L + u \circ L = L + L \circ u + \varphi(I+u)$

$u \circ L - L \circ u = \varphi(I+u)$

Let $\mathcal{L}: C_b(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$ be defined as

$\mathcal{L}(u) = u \circ L - L \circ u$

Lemma 3: $\mathcal{L}$ is invertible, $\|\mathcal{L}^{-1}\| \leq \frac{\|L^{-1}\|}{1-a}$

(a - was defined in lemma 1, the proof of this lemma refers to lemma 2)

we search for u satisfying $u = \overbrace{\mathcal{L}^{-1}(\varphi(I+u))}^{\psi}$

observation: the map $\psi: C_b(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$ is a contraction if ε was small

proof of observation: u_1, u_2

$\|\psi(u_1) - \psi(u_2)\| \leq \|\mathcal{L}^{-1}\| \, \|\varphi(I+u_1) - \varphi(I+u_2)\| \leq \varepsilon \|u_1 - u_2\|_\infty \|\mathcal{L}^{-1}\|$

01. 04. 2011

Proof (finish): $g \approx f$, $\mathbb{R}^2$, $G \approx T$ C^1 close

$G = T + \varphi$, φ - C^1 close to 0 (in particular Lipschitz continuous with L constent $< \varepsilon$)

we have to cheach that $\exists\, \varepsilon > 0$ so that $f \approx g$ (ε - close in C^1 - topology)
then $\exists$ $\mathbb{T}^2 \circlearrowleft$ homeo

$$\begin{tikzcd}
\mathbb{T}^2 \arrow[r, "f"] \arrow[d, "h"'] & \mathbb{T}^2 \arrow[d, "h"] \\
\mathbb{T}^2 \arrow[r, "g"] & \mathbb{T}^2
\end{tikzcd}$$

proof the same as G - H thm

Proof of G - H thm . Main step: $L + \varphi$

$H = I + u,$ $\underbrace{(I+u)}_{H} \circ L = (L+\varphi)(I+u)$ $(*)$

we have found $u \in C_b(\mathbb{R}^m \to \mathbb{R}^m)$ which satisfies $(*)$. How to cheack that H is homeo ?

Actually , insted of $(*)$, one could have proved $(**)$
$\exists\, \varepsilon > 0$ so that if φ_1, φ_2 are ε - close to 0 in $C_b(\mathbb{R}^m \to \mathbb{R}^m)$
then there exists a unique solution of $(I+u) \circ (L+\varphi_1) = (L+\varphi_2)(I+u)$

How to cheack that H is homeo ?
Using $(**)$ we produce a unique $\overbrace{v \in C_b(\mathbb{R}^m \circlearrowleft)}$ satysfying $(I+v) \circ (L+\varphi) = L(I+v)$ $(I+u)(I+v)$
so we have $(I+u)(I+v) = I$
similarily $(I+v)(I+u) = I$, we use uniqueness of solution

3 lemmas used in the proof

1st
2nd

Lemma 3 $\mathcal{L} : C_b(\mathbb{R}^m) \to C_b(\mathbb{R}^m)$
$\mathcal{L}(n) = n \circ L - L \circ n$ is invertible $\mathbb{R}^n = E^s \oplus E^n$
hyperbolic , linear operator

Proof: $C_b(\mathbb{R}^m \to \mathbb{R}^m) = C_b^s \oplus C_b^u$
so $u = u^s + u^u$

$\mathcal{L}$ preserves both C_b^s, C_b^u
we can show $\mathcal{L}$ invertible on both of them
$\mathcal{L}|_{C_b^s} = L(L^{-1} \circ u \circ L - u)$
$\mathcal{L}|_{C_b^u} = L(L^{-1} \circ u \circ L - u)$ $-(I - L^{-1} \circ (\cdot) \circ L)$
$u \mapsto L^{-1} \circ u \circ L$ - contraction in C_b^u
$I +$ sth of small norm is invertible
|
L is contraction here

The proof of proposition is finished. we have proved that if $L : \mathbb{R}^n \circlearrowleft$ is a hyperbolic linear operator then there exists $\varepsilon > 0$ such that in $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ $\varphi \in C_b(\cdot)$ and $\mathrm{Lip}(\varphi) < \varepsilon$, then there exist a homeomorphism H : $H \circ L = (L+\varphi) \circ H$
we used the proposition to finish the proof of a structural stability of a hyperbolic automorphism of $\mathbb{T}^2$ $f : \mathbb{T}^2 \circlearrowleft$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "L"] \arrow[d] & \mathbb{R}^2 \arrow[r, shift left, "H"] \arrow[d] & \mathbb{R}^2 \arrow[l, shift left] \arrow[r, "G"] \arrow[d] & \mathbb{R}^2 \arrow[d] \\
\mathbb{T}^2 \arrow[r, "f"] & \mathbb{T}^2 \arrow[r, shift left, "h"] & \mathbb{T}^2 \arrow[l, shift left] \arrow[r, "g"] & \mathbb{T}^2
\end{tikzcd}$$

$G = L + \varphi$

The proposition gives us a homeomorphism $H :$ $G \circ H = H \circ L$

Observation: If $H = I + u$ and u is doubly periodic (i.e) $u(x+p, y+q) = u(x,y)$ $p, q \in \mathbb{Z}^2$ then H gives a homeomorphism $h : \mathbb{T}^2 \circlearrowleft$
$h = \pi \circ H \circ \pi^{-1}$

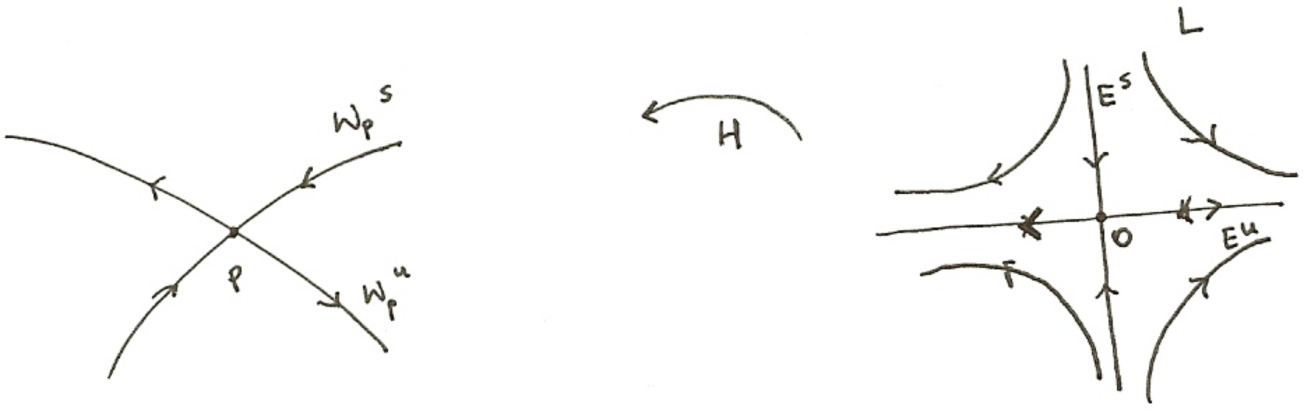
<u>Lemma:</u> If $G = L + \varphi$ and φ is doubly periodic then the solution u of the equation

$$(I + u) \circ L = (L + \varphi) \circ (I + u)$$ is also doubly periodic

This follows from the fact that the operator

$$\mathcal{L} : C_b(\mathbb{R}^m) \longrightarrow C_b(\mathbb{R}^m)$$

preserves the subspace of $C_b^{dp}(\mathbb{R}^2 \to \mathbb{R}^2)$
doubly periodic



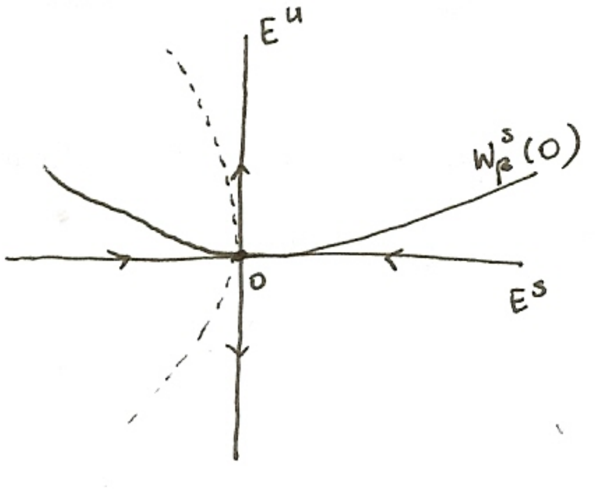
<u>Theorem:</u> F - diffeomorphism C^r in neighbourhood of 0 in $\mathbb{R}^n$, $F(0) = 0$, $DF(0) = L$ hyperbolic linear ~~map~~ automorphism. Then there exists $\beta > 0$ such that $W_\beta^s(0) = \{x : \forall_n \ f^n(x) \in \cancel{B} B_\beta(0)\}$

$$W_\beta^u(0) = \{x : \forall_n \ f^{-n}(x) \in B_\beta(0)\}$$

are C^r regular submanifolds of $B_\beta(0)$. Moreover $T_0(W_\beta^s(0)) = E^s$
$T_0(W_\beta^u(0)) = E^u$

<u>Generalization:</u> If $F : M \circlearrowleft$ is a diffeomorphism of a smooth manifold and p is a hyperbolic fixed point, then the same holds for $W_\beta^s(p)$, $W_\beta^u(p)$

<u>Proof:</u>

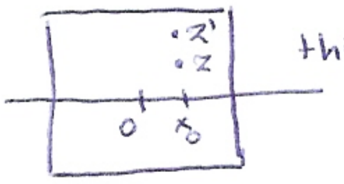


8.04.2011

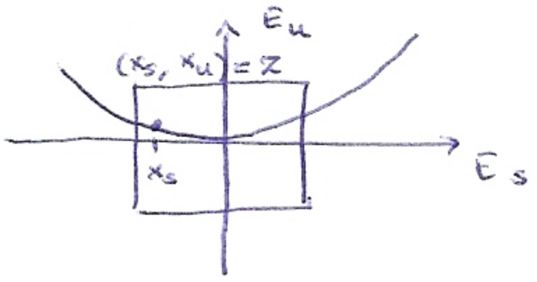
Hadamard-Perron theorem

Proof (without details)

Lemma 1: if $z = (x_s, x_u) \in B_\beta(0)$ and $z' = (x_s, x_u') \in B_\beta(0)$ and both z, z' are in $W_\beta^s(0)$ then $z = z'$



this cannot happen



our propose: given x_s $(x_s \approx 0)$, to find x_n so that the point $z = (x_s, x_n) \in W_\beta^s(0)$

we introduce the space

$$K = \{ \gamma(n) : n \geq 0 \ : \gamma(n) \in \mathbb{R}^m \ : \gamma(n) \to 0 \text{ as } n \to \infty \}$$

$$\| \gamma(n) \| = \sup_n \| \gamma(n) \|_{\mathbb{R}^m}$$

Proposition: (left as an exersize) K is complete so it is a Banach space.

Let $G \subset K = \{ \gamma(n) : \text{such that } \gamma(n) \in B_\beta(0) \text{ for all } n \}$

Given a point $z \in W_\beta^s(0)$, we can take the sequence $\gamma(n) = F^n(z) \in K$

Then

$$\gamma(n) = F^n(z) = (D+f)^n(z) = (D+f) \circ (\underbrace{(D+f)^{n-1}(z)}_{\gamma(n-1)}) = D((D+f)^{n-1}(z)) + f(\gamma(n-1))$$

$$= D((D+f)(D+f)^{n-2}(z)) + f(\gamma(n-1)) = D^2((D+f)^{n-2}(z)) + D(f(\gamma(n-2)) + f(\gamma(n-1)) =$$

$$= \dots = D^n(z) + \sum_{i=0}^{n-1} D^{n-i-1} f(\gamma(i))$$

D preserves E^u and E^s

$\gamma(n) = \gamma^u(n) + \gamma^s(n)$

$x = z_u + z_s$

$$(*) \quad \gamma^u(n) = (D^u)^n(z_u) + \sum_{i=0}^{n-1} D^{n-i-1} f^u(\gamma(i)) = (D^u)^n (z_u + \sum_{i=0}^{n-1} (D^u)^{-1-i} (f^u(\gamma(i)))).$$

$f = f^u + f^s$ Remember that $\gamma(n) = \gamma^s(n) + \gamma^u(n)$ is a trajectory of a point $z \in W_\beta^s(0)$ $\Rightarrow$ both $\gamma^s(n)$ and $\gamma^u(n)$ must ~~tends~~ be bounded ~~to 0~~ as $n \to \infty$. This implies that

$$z_u * = - \sum_{i=0}^{\infty} (D^u)^{-1-i} (f^u(\gamma(i))) = \gamma^u(0) \quad (**)$$

Putting it into $(*)$ we get the formula for $\gamma^u(n)$

$$\gamma^u(n) = - \sum_{i=n}^{\infty} (D^u)^{n-i-1} f^u(\gamma(i)) \quad (***)$$

$$\gamma^s(n) = (D^s)^n(\overset{s}{z}) + \sum_{i=0}^{n-1} (D^s)^{n-i-1} f^s(\gamma(i)) \quad (****)$$

Now we define a map $G : \underset{E^s \supset}{B_\beta^s} \times G \longrightarrow K$

$$\underset{\underset{B^s_\beta}{\cap}}{G(x,}\ \underset{\underset{\mathcal{G} \subset K}{|}}{\gamma)}(n) = \gamma(n) - [(* * * *), (* * *)] =: \tilde{\gamma}(n)$$

$$= \gamma(n) - \left[(D^s(x))^n + \sum_{i=0}^{n-1} D^{n-i-1} f^s(\gamma(i)),\ - \sum_{i=n}^{\infty} (D^u)^{n-i-1} f^u(\gamma(i))\right]$$

Strategy: to solve the equation $G(x,\gamma) = 0$ and to verify that if $G(x,\gamma) = 0$ $\gamma(n)$ is a trajectory $F^n(z)$ of some point $z = (x, x_u)$ and $z \in W^s_\beta(0)$

$\tilde{\gamma}(n) \xrightarrow{n\to\infty} 0$

we claim that the function $G(x,\gamma)$ is C^1- smooth

we shall calculate

$$\frac{\partial G}{\partial \gamma}(x,\gamma)(u)(n)$$

$u(n) \in K$

$$\frac{\partial G}{\partial \gamma}(x,\gamma)(u)(n) = u(n) - \left[\sum (D^s)^{n-i-1} df^s(\gamma(i))(u(i)),\ - \sum_{i=n}^{\infty} (D^u)^{n-i-1} df^u(\gamma(i))(u(i))\right]$$

$$\frac{\partial G}{\partial \gamma}(0,0)(u(n)) = u(n)$$

we conclude that there exists a C^1 map (defined in a ngbh of 0 in E^s)

$x \xrightarrow{\varphi} (\gamma(n))(x)$ such that

$$G(x, \varphi(x)) = 0$$

In particular, we have $\gamma(0) = [x,\ - \sum_{i=0}^{\infty} (D^u)^{-1-i} f^u(\gamma(i))]$

$(x, \gamma^u(0)) \in B^s_\beta(0)$

Define a function h as a composition

$$\begin{tikzcd}
x \arrow[r, "C^1"] \arrow[rrr, bend right, "C^1"'] & \underset{\overset{\shortparallel}{\gamma(n)}}{\varphi(x)} \arrow[r, "\vartheta", "\text{projection onto ``0'' coordinate}"'] & \gamma(0) \arrow[r, "\pi_2", "\text{projection onto } E^u \text{ coordinate}"'] & (\gamma(0))^u
\end{tikzcd}$$

$(x, h(x)) \in W^s_\beta$ $\qquad D_0 h : E^s \longrightarrow E^u$

$G(x, \varphi(x))$

$$\partial_x G + \partial_\gamma G \circ \partial_x \varphi = 0$$

$$\partial_x \varphi = - \underset{id}{[\partial_\gamma G]^{-1}} \circ \partial_x G$$

~~$\frac{\partial G}{\partial x}(0,0)(v) = (v, D^s(v), (D^2)^2(v), \dots)$~~

$$\frac{\partial G}{\partial x}(0,0)(v)(n) = [(D^s)^n(v), 0] \xrightarrow{D\vartheta} [v, 0] \xrightarrow{\pi_2} 0$$

15.04.2011

Complement of the proof of G-H theorem

we have proved it for diffeomorphisms defined in a neighbourhood of 0. $F = L + f$, L- hyperbolic

Usually G-H is formulated for flows

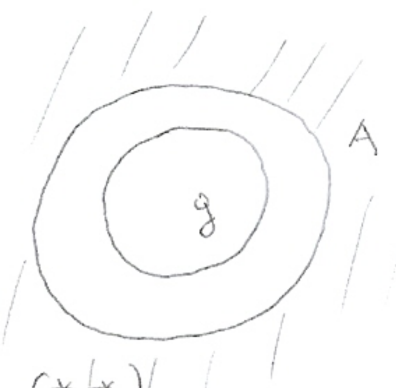
(*) $x' = g(x)$, $g(0) = 0$, $Dg(0) = A$ has no eigenvalues with $\mathrm{Re}\,\lambda = 0$

φ_t flow of (*) is conjugete to the linear flow $\dot{x} = Ax$ in some neighbourhood of 0.

Proof for flows: This follows from the proof for diff:

, we modify g to $\tilde{g}$ so that

$$\tilde{g} = \begin{cases} g & \text{in a small neighbourhood of } 0 \\ A & \text{outside a larger neighbourhood of } 0 \end{cases}$$



Let $\tilde{\varphi}_t$ be a flow for $\tilde{g}$

$\tilde{\varphi}_t$ is the globally defined (for all t)

(follows from Gronwall inequelity)

Using the proven part of G-H, we have

Proposition: There exists a homeo H $\overbrace{\tilde{\varphi}^1 \circ H = H \circ e^A}^{(**)}$ ($\tilde{\varphi}^1 = e^A +$ the rest)

we expect that this homeomorphism H conjugates in the whole flows, i.e. $\tilde{\varphi}_t \circ H = H \circ e^{At}$

why?

Fix t and consider the new homeomorphism $H_t = \tilde{\varphi}^t \circ e^{-At}$

Then (check!) H_t also conjugates - satisfies (* *)

Thus if we check that H_t has a form $\boxed{H_t = I + u,}$ (* * *)

$u \in C_b(\mathbb{R}^n)$ the uniqueness of the conjugacy (see the proof of G-H for diff) will imply $H_t = H$, $\tilde{\varphi}^t \circ H \circ e^{-At} = H$

$$(H_t - I) = \tilde{\varphi}^t \circ H \circ e^{-At} - e^{At} \cdot e^{-At} = \underbrace{(\tilde{\varphi}^t - e^{At}) \circ H \circ e^{-At}}_{\text{bounded}} + \underbrace{e^{At}(H - I)e^{-At}}_{\text{bounded}}$$

t-fixed

Conclusion: $H_t = I + u$, u- bounded. (* * *) are checked

To finish the proof we have to check that the assumptions of G-H thm (for diff) are satisfied for $\tilde{\varphi}^1$ (to formulate the Proposition)

Pani poleca suity Bacha na wiolonczelę.

$$\frac{d}{dt}\tilde{\varphi}^t(x) = \underbrace{(A + \tilde{r})}_{\tilde{g}}\tilde{\varphi}^t(x) \qquad \dot{x} = \tilde{g}(x)$$

$$\frac{d}{dt}(e^{-At}\tilde{\varphi}^t(x)) = -Ae^{-At}\tilde{\varphi}^t(x) + e^{-At}(A + \tilde{r})\tilde{\varphi}^t(x) = e^{-At}\tilde{r}\tilde{\varphi}^t(x)$$

so $e^{-At}\tilde{\varphi}^t(x) = \int_0^t e^{-As}\tilde{r}(\tilde{\varphi}^t(x))\,ds + x$

$$\tilde{\varphi}^t(x) = e^{At}x + \int_0^t e^{A(t-s)}\tilde{r}(\tilde{\varphi}^t(x))ds$$

$$\tilde{\varphi}^1(x) = e^{A}x + \int_0^1 e^{A(1-s)}\tilde{r}(\tilde{\varphi}^1(x))ds$$

$F = Lx + f$ f has a small Lipschitz constant (?)

$$\tilde{r}(\tilde{\varphi}^1(x)) - \tilde{r}(\tilde{\varphi}^1(x')) \leq (Lip)\, \|\tilde{\varphi}^1(x) - \tilde{\varphi}^1(x')\| \leq c \cdot \|x - x'\|$$

Def: Global stable (unstable) manifold of a hyperbolic fixed point

$W^s_p \overset{(f)}{\cancel{(X)}} = \{x : f^n(x) \to p\}$ $f: M \circlearrowleft$ at least C^1-diff

$= \{ \bigcup_{n \geq 0} f^{-n}(W^s_{p,loc}) \}$

Def': For flows X - C^1 - vector fields of M, $\dot{x} = X(x)$ diff. eq. related to this field, p - stacionary, hyperbolic point ($(X(p)) = 0$ (it implies that p is a fixed, hyperbolic point of φ^t)

$W^s_p(X) = \{x : \varphi^t(x) \longrightarrow p\}$.

Proposition: $W_p^s(\varphi^1) = W_p^s(\varphi^t)$

Proposition (Excersizes): $W_p^s(f)$ $(W_p^s(X))$ is an immersed submanifold (of the same class C^r as f $(X'))$

Def: If γ - periodic orbit with period T of a flow X then γ is called hyperbolic if $D\varphi^T(x)$ has only one (simple) eigenvalue with modules 1

Def: If γ is a hyperbolic, periodic orbit then

$W^s_\gamma = \{z : \omega(\{\varphi^t(z)\}) = \gamma$

Picture (Ω - explosion) but first

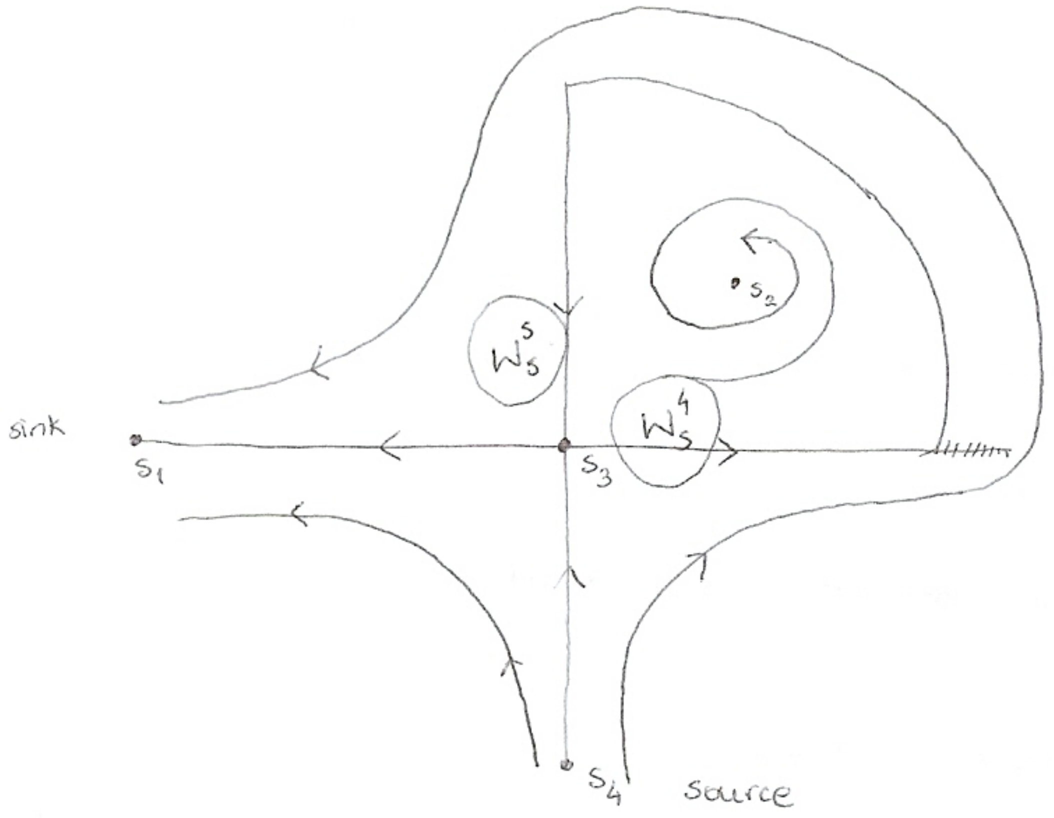
Def: $\Omega(f)$

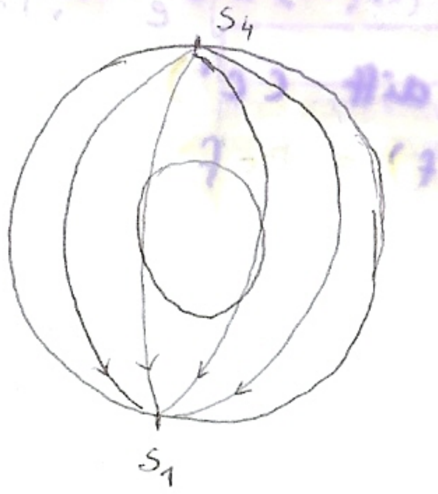
$\Omega(X)$ X flow

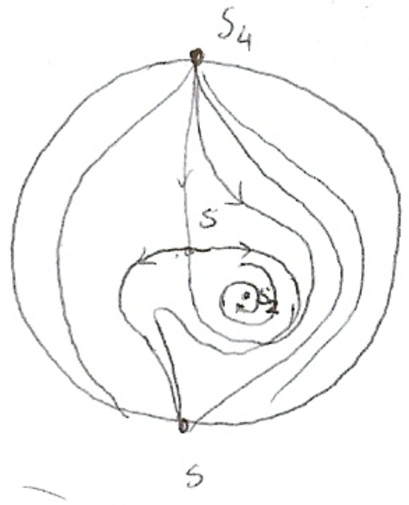
wondering point for a flow: x is wondering if $\exists\, t_0$ and a neighbourhood $U \ni \{x\}$ such that

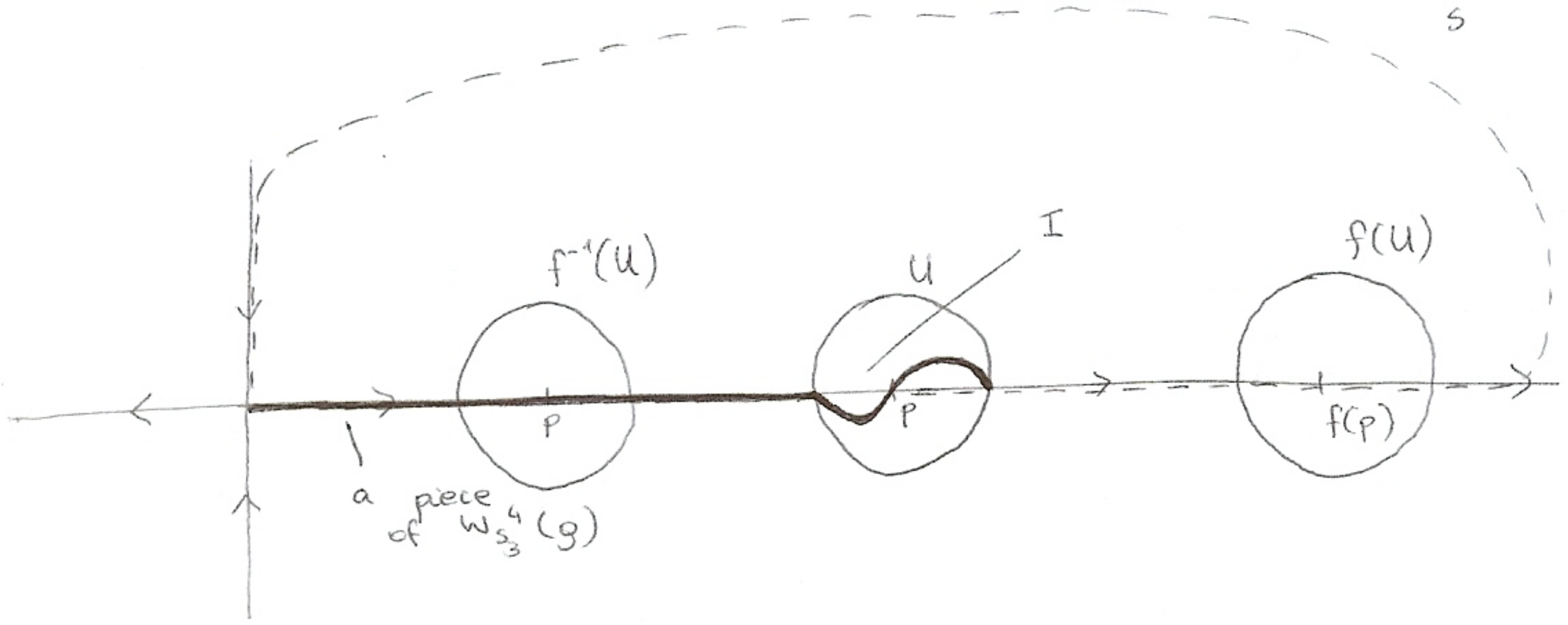
$\varphi^t(\bar{U}) \cap U = \emptyset$

for all $|t| > t_0$









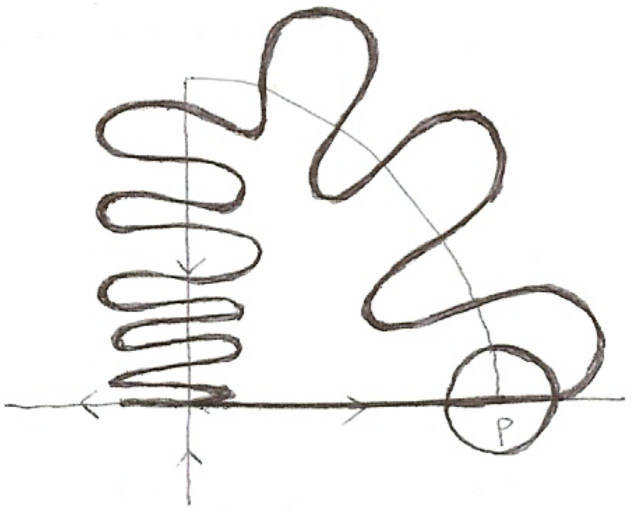
Let i be defined so that $i = id$ outside U

we modify f (only locally) and $i(I) =$ 

we put

$$g = i \circ f$$

$W^4_{s_3}(g)$, $W^s_{s_3}(g)$



29. 04. 2011

Morse Smale systems:

(we know what it means for diffeos on S^1 and we know that M-S diff $S^1 \circlearrowleft$ form an open set, dense subset and these are the only structurally stable diff $S^1 \circlearrowleft$)

Definition:

$f: M \circlearrowleft$ diff (C^1) is Morse - Smale iff

1) $\Omega(f) = \mathrm{Per}\, f$
2) $\# \mathrm{Per}\, f < \infty$
3) all periodic points are hyperbolic
4) $W^s(p) \cap W^u(q)$ intersect transversally (or not at all)

Def:

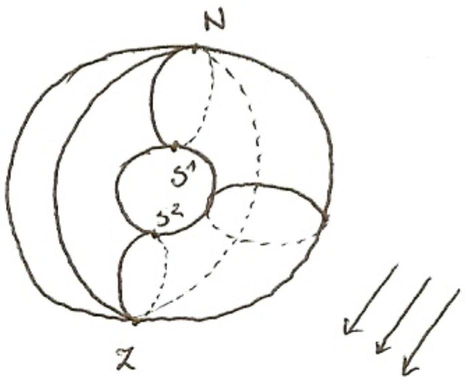
X - vector field on a compact, smooth manifold M.
φ^t - flow of X.
X is called Morse - Smale vector field iff

1) $\Omega(X)$ is a union of finite number of stationary points $(X=0)$ or periodic orbits.
2) all these points and orbits are hyperbolic (called critical elements)
3) If σ_1, σ_2 are critical elements then $W^s(\sigma_1), W^u(\sigma_2)$ intersect transversally.

If γ is a periodic orbit then $W^s(\gamma) = \{ y : \omega(y) = \gamma \}$

Theorem: This is an immersed C^1 manifold

Example 1: Consider torus; $h(z) = -z$, $X = \mathrm{grad}\, h$.



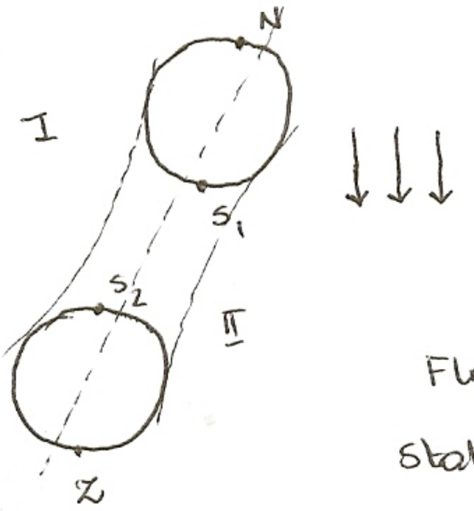
$v \in T_pM$: $Dh(v) = \langle \mathrm{grad}_M h, v \rangle$

$\mathrm{grad}_M h$ = orthogonal projection of $\mathrm{grad}\, h$ to T_pM

No M-S system - two saddles connect (stable and unstable don't intersect transversally)

Example 2:

"Lean" the torus. Look at section of this torus.



If the flows starts from part II it remains in par II

There is no more connection between $W^u(S_1)$ and $W^s(S_2)$.

Flow from example I is not structurally stable, because after a small perturbation the connection between saddles S_1, S_2 disappeared.

Flow from example 2 is M-S flow.

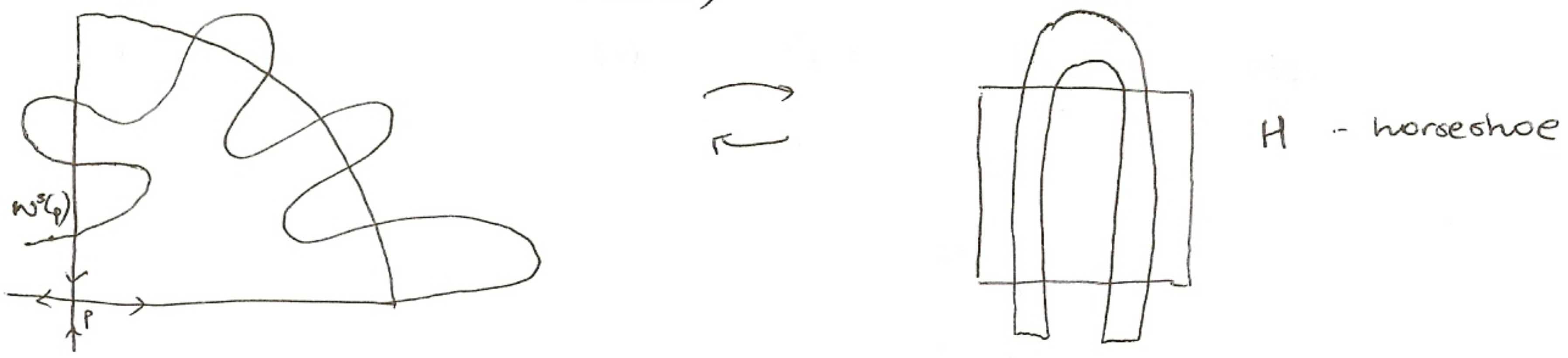
__Theorem:__ M - S diffeomorphism from an open set in Diff (M).
Every M - S diff is structurally stable.

__Thm:__ M - S vector fields form an open set in the space of C^1 - vector fields on M. Each M - S vector field is structurally stable.

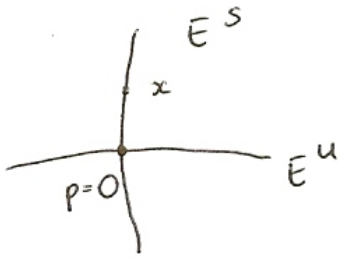
__Warning:__ M - S systems are not dense and there are many other structurally stable diff and flows.
ONLY on S^1 M - S <=> structurally stable

Last time we almost discovered that M - S diff on S^2 cannot be dense, since we discovered homoclinic points

__Thm:__ If $f : M \circlearrowleft$ is a C^1 - diff, p - hyperbolic fixed point and there is a homoclinic point x (i.e. $W^s(p)$, $W^u(p)$ intersect transversally at p). Then there exists a hyperbolic invariant set Λ such that the dynamics of $f^N|_\Lambda$ is conjugate to $H|_{\Omega(H)}$



__Proof:__ (outline)



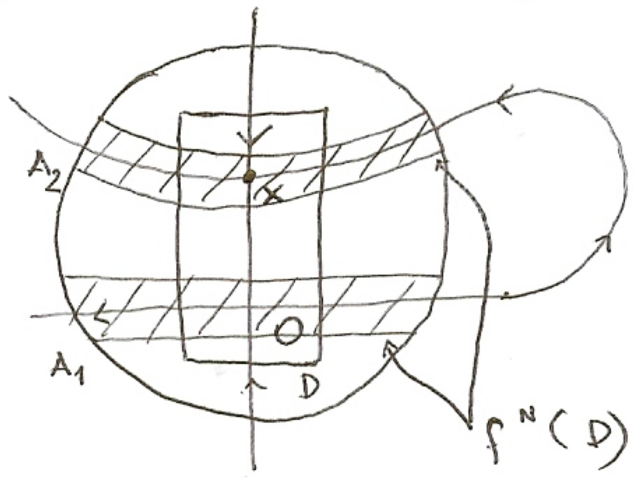
$$(\underset{\underset{E^u}{\cap}}{x}, \underset{\underset{E^s}{\cap}}{y}),$$

$$W^u_\varepsilon(p) = (x, \varphi^u(x))$$

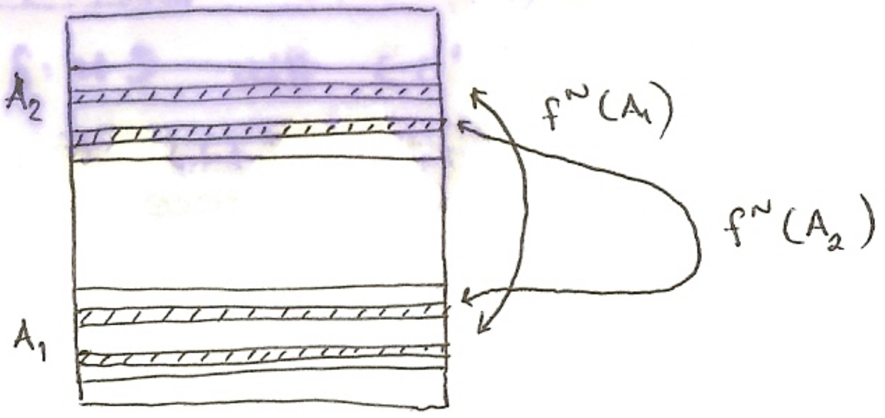
$$W^s_\varepsilon(p) = (\varphi^s(y), y)$$

Change of coordinates

$$\varphi(x,y) = (x - \varphi^s(y), y - \varphi^u(x))$$



we claim, that for some N the intersection $f^N(D) \cap D$ has two components A_1, A_2.



The precise calculation is based on so called λ- lemma (or Inclination lemma)

p- hyperbolic fixed point, $W^u_\varepsilon(p)$ - a "piece" of stable manifold or $f^M(W^u_\varepsilon(p))$ for some M.

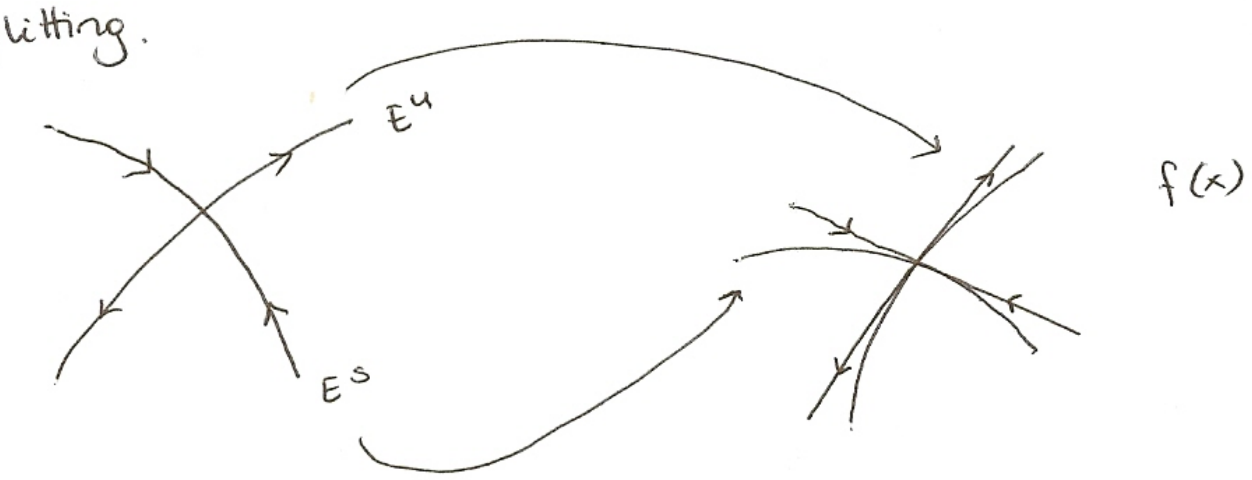
If D^u is a small disc (n- dimensional), transversal to $W^s(p)$ at x then for all $N >> 0$.

$f^N(D^u)$ contains an open subset which is ε- C^1- close to $W^u_\varepsilon(p)$

Recall: $f: M \circlearrowleft$ diff, M - compact, we may assume that $f: \mathcal{U} \to M$ for some nbd of Λ

Λ - hyperbolic set - compact, f- invariant

Recall, that at each point $x \in \Lambda$ there is a invariant splitting.



Theorem: (on existence of $W^s_\varepsilon(x)$, $W^u_\varepsilon(x)$ for hyperbolic sets)

If Λ is h.s. then there exists $\varepsilon, \tilde{\lambda}$ s.t. for every $x \in \Lambda$:

$$W^s_\varepsilon(x) = \{ y : d(f^n(x), f^n(y)) < \varepsilon \quad \forall \quad n \geq 0 \} =$$

$$= \{ y : d(f^n(x), f^n(y)) < C \tilde{\lambda}^{-n} \text{ for all } n \geq 0 \}$$

is a smooth manifold, tangent to $E^s(x)$ at x.

Moreover $f(W^s_\varepsilon(x)) < W^s_\varepsilon(f(x))$

13.05.2011

Measure preserving transformation:

$(X, \mathfrak{M}, \mu)$ X - a set, $\mathfrak{M}$ - a field of subsets of X, μ - measure, (σ - field)
μ - finite

$T: X \to X$, we always assume that T is μ-measurable, i.e.
$\forall A \in \mathfrak{M}$ $T^{-1}(A) \in \mathfrak{M}$.

Def: we say that T preserves the measure μ (or μ is T-invariant)
if $\forall A \in \mathfrak{M}$ $\mu(T^{-1}(A)) = \mu(A)$ (T-endomorphism of X)

Remark: If T is invertible, then μ is T-invariant $\iff$ $\mu(T(A)) = \mu(A)$
(T-automorphism of X)

Fact: T preserves μ iff for every function $f: X \to \mathbb{R}$, $f \in L_1(\mu)$
$\int f \circ T \, d\mu = \int f \, d\mu$ (*)

Proof: If $f = \chi_A$ for $A \in \mathfrak{M}$, then
$\int f \circ T \, d\mu = \int \chi_A(T(x)) \, d\mu(x) = \int \chi_{T^{-1}(A)}(x) \, d\mu(x) = \mu(T^{-1}(A))$
$\int f \, d\mu = \int \chi_A \, d\mu = \mu(A)$
This gives the proof of „$\Leftarrow$"

$\Rightarrow$ we have (*) for $f = \chi_A$ $A \in \mathfrak{M}$
Then the standard proof from integration theory
indication functions $\Rightarrow$ simple function $\Rightarrow$ measurable - positive functions
$\Rightarrow$ L^1 functions

Remark: $U(f) = f \circ T$ $\leftarrow$ operator on $L^1(\mu)$
$L^2(\mu)$
(Koopman operator)

(spectral theory)

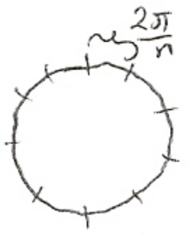
Examples of measure preserving transformations:

1) Rotation of the circle - it preserves Lebesgue measure, T-automorphism
2) Rotation of the torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$

$(x_1, \dots, x_n) \longmapsto (x_1 + \alpha_1, \dots, x_n + \alpha_n)$ mod 1 $\alpha_1, \dots, \alpha_n \in \mathbb{R}$

(generalization: the Haar measure on a locally compact group)

(3) $X = S^1 \subset \mathbb{C}$
$T(z) = z^n$, $n \in \mathbb{Z} \setminus \{0\}$
$T(x) = nx$ mod 1, $x \in [0,1]$



each arch goes to the full circle

Lebesgue measure preserved

4) Gauß transformation
$T: (0,1] \to (0,1)$ $T(x) = \frac{1}{x} - [\frac{1}{x}] = \{\frac{1}{x}\}$
- fraction part of $\frac{1}{x}$

It is well defined and values are in the set $[0,1]$

8) Liouville theorem: If M - smooth manifold, V - smooth vector field on M, $\{\varphi_t\}$ flow defined by M.
If $\operatorname{div} V = 0$ then φ_t preserves Lebesgue measure on M.

<u>Poincare recurrence theorem</u> (tw Poincarego o powracaniu) (1899)
$(X, \mathfrak{M}, \mu)$ - measurable space, $T: X \to X$, μ - finite

<u>Def</u>: A point $x \in X$ is recurrent w.r.t. a set $A \in \mathfrak{M}$ if $\exists n > 0$ s.t. $T^n(x) \in A$

<u>Theorem</u> (PRT) If T preserves μ, then for every $A \in \mathfrak{M}$, μ - -almost every point of A is recurrent with respect to A.

<u>Corollary</u>: μ - a.e. point of A returns to A infinetly many times

<u>Proof</u>: $B = \{x \in A \ \ \forall n>0 \ \ T^n(x) \in A\}$

Let $B_n = T^{-n}(B)$
Then B_n are pointwise disjoint. Because if $x \in B_k \cap B_l$ $k > l$
$x \in T^{-k}(B) \cap T^{-l}(B)$

$$\underset{\substack{\rotatebox{90}{$\in$} \\ B \\ \cap \\ A}}{T^{k}(x)} = T^{k-l}(\underset{\substack{\rotatebox{90}{$\in$} \\ B}}{\underbrace{T^{l}x}}) \quad \text{it's contradiction}$$

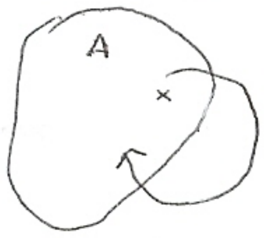
Moreover, since μ is T invariant $\forall n \ \mu(B_n) = \mu(B)$.

$$\underset{\substack{\wedge \\ \mu(X) < \infty}}{\mu(\bigcup_{n=1}^{\infty} B_n)} = \sum_{n=1}^{\infty} \mu(B) \quad \text{so} \quad \mu(B) = 0$$

▨

<u>Def</u>: $A \in \mathfrak{M}$, $\mu(A) > 0$
$x \in A$, $n_A(x) = \inf\{n > 0 : T^n(x) \in A\}$
by PRT, n_A is well-defined almost everywhere on A
n_A - first return time to A

$T_A : A \to A$ the first return map to A (induced map) ← it's well defined almost everywhere on A
$T_A(x) = T^{n_A(x)}(x)$



T_A preserves μ
Kakutani diagram (tower) ~ 1840

$A_k = \{x \in A : n_A(x) = k\}$ $k = 1, 2, \dots$
$A = \bigcup_{k=1}^{\infty} A_k$ (up to zero-measure set)

8) Liouville theorem : If M - smooth manifold , V - smooth vector field on M, $\{\varphi_t\}$ flow defined by M.
If $\operatorname{div} V = 0$ then φ_t preserves Lebesgue measure on M.

Poincare recurrence theorem (tw Poincarego o powracaniu) (1899)
$(X, \mathfrak{M}, \mu)$ - measurable space, $T: X \to X$, μ - finite

Def: A point $x \in X$ is recurrent w.r.t. a set $A \in \mathfrak{M}$ if $\exists n > 0$ s.t. $T^n(x) \in A$

Theorem (PRT) If T preserves μ, then for every $A \in \mathfrak{M}$, μ - -almost every point of A is recurrent with respect to A.

Corollary: μ - a.e. point of A returns to A infinetly many times

Proof: $B = \{x \in A \quad \forall n > 0 \quad T^n(x) \notin A\}$

Let $B_n = T^{-n}(B)$

Then B_n are pointwise disjoint. Because if $x \in B_k \cap B_l$ $k > l$
$x \in T^{-k}(B) \cap T^{-l}(B)$

$$\underset{\overset{\cap}{\underset{\hat{A}}{B}}}{T^k(x)} = T^{k-l}(\underset{\overset{\cap}{B}}{\underbrace{T^l x}})$$ it's controdiction

Moreover, since μ is T invariant $\forall n \; \mu(B_n) = \mu(B)$.

$$\underset{\overset{\wedge}{\mu(X) < \infty}}{\mu(\bigcup_{n=1}^{\infty} B_n)} = \sum_{n=1}^{\infty} \mu(B) \quad \text{so} \quad \mu(B) = 0$$

▨

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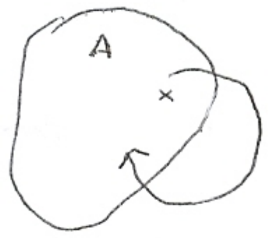

T_A preserves μ

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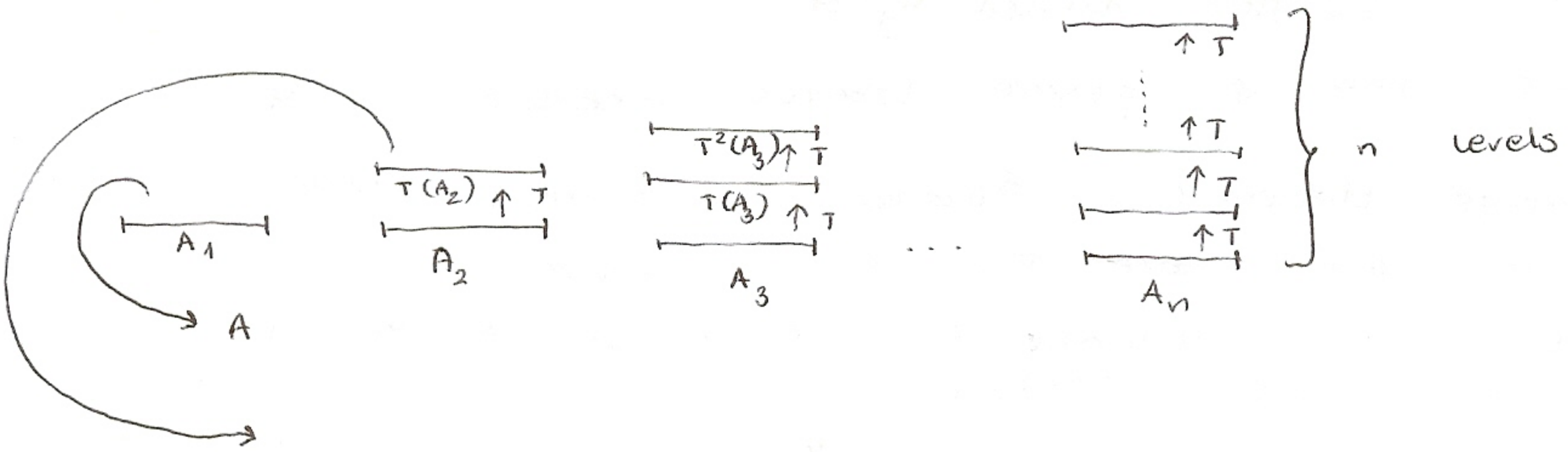


$T(A_2) = (TA \setminus A)_2$
$T(A_3) = (TA \setminus A)_3$
$T^2(A_3) = (T^2A \setminus A)$

Dynamics on the tower is given by arrows. Tower is forward invariant ($T(I) \subset I$ by construction).

Note that $\underset{\substack{\uparrow \\ \text{tower}}}{T^{-1}(J) \setminus J}$ has measure 0. So we can neglect it:

(Points from $T^{-1}(J) \setminus J$ never return to $T^{-1}(J) \setminus J$, use PRT)

<u>Corollary:</u> The first return map - T_A preserves measure $\mu|_A$

<u>Proof:</u> Looking at the tower

<u>Ergodic theorems:</u> $(X, \mathfrak{M}, \mu)$ - measurable space, μ- finite, $T: X \to X$ preserves μ.

1. Maximal ergodic theorem (wierer - Yosida - Kakutani 1939)

$\forall f \in L^1(\mu)$ $f: X \to \mathbb{R}$ if $S_n(x) = f(x) + f(T(x)) + f(T^2(x)) + \dots + f(T^{n-1}(x))$

then $\int\limits_{\{x \in X: \sup\limits_n S_n(x) > 0\}} f \, d\mu \geq 0$

<u>Proof:</u> $A = \{x : \sup\limits_n S_n(x) > 0\}$, $S_n(x) = \max(S_1(x), \dots, S_n(x))$

$A_n = \{x : S_n(x) > 0\}$, $S_n^*(x) = \max(S_n(x), 0)$

By definition $S_{n+1}(x) = f(x) + S_n^*(Tx)$

(if $S_n^*(Tx) = S_n(Tx)$ then $R = f(x) + S_k(Tx) = f(x) + f(Tx) + \dots + f(T^k(x)) = 1$
$1 \leq k \leq n$

if $S_n^*(Tx) = 0$ then $L = R = f(x)$)

$$\int\limits_{A_n} f = \int\limits_{A_n} (S_{n+1} - S_n^* \circ T) \geq \int\limits_{A_n} (S_n - S_n^* \circ T)$$

$$\int\limits_{A_n} f = \int\limits_{A_n} (S_{n+1} - S_n^* \circ T) \underset{S_{n+1} \geq S_n}{\geq} \int (S_n - S_n^* \circ T) \underset{\substack{\uparrow \\ S_n^* = S_n \\ \text{on } A_n}}{=} \int\limits_{A_n} (S_n^* - S_n^* \circ T) =$$

$$= \int_X s_n^* - \int_{A_n} s_n^* \circ T \geq \int_X s_n^* - \int_X \underbrace{s_n^*}_{\text{It's non-negative}} \circ T = 0$$

so $\int_A f = \int_{\bigcup_{n=1}^{\infty} A_n} f \geq 0$ ▨

Theorem (Birkhoff ergodic theorem) 1931

$\forall \; f \in L^1$

a) $\frac{1}{n} f(x) + f(T(x)) + \ldots + f(T^{n-1}(x)) \xrightarrow[n \to \infty]{} \bar{f}(x)$ μ almost everywhere and in L^1

b) $\bar{f}$ T- invariant , $\bar{f} \in L^1$, $\|\bar{f}\|_1 \leq \|f\|_1$

c) $\int_A \bar{f} \, d\mu = \int_A f \, d\mu$ $\forall$ T- invariant set , $A \in \mathfrak{M}$ $(A = T^{-1}(A))$

$\bar{f} = \mathbb{E}(f \mid \mathcal{G})$ $\mathcal{G}$ = σ- field of T- invariant set.

Proof: a) $S_n(x) = f(x) + \ldots + f(T^{n-1}x)$

Suppose $\frac{S_n(x)}{n}$ does not converge on a set of positive measure.

Then $\underline{\lim} \frac{S_n(x)}{n} < b < a < \overline{\lim} \frac{S_n(x)}{n}$ for $a, b \in \mathbb{R}$ $\forall x \in A$, $\mu(A) > 0$

Since A is compl. invariant we can assume $A = X$.
we can rescale f taking $\alpha f + \beta$ instead of f $(\alpha, \beta \in \mathbb{R})$, we can assume that $a = 1$, $b = -1$

so $\overline{\lim} \frac{S_n(x)}{n} > 1$ $\forall x \in X$ Then $\sup_n S_n(x) > 0$

$\underline{\lim} \frac{S_n(x)}{n} < -1$ by MET $\int_X f \geq 0$

Take $-f$ instead of f. Then by MET $\int_X f \leq 0$ so
$\int_X f = 0$.

Take $f + \frac{1}{2}$ instead of f then for suitable $S_n(x)$ we have
$\overline{\lim} \frac{S_n(x)}{n} > \frac{3}{2}$ $\underline{\lim} \frac{S_n(x)}{n} < -\frac{1}{2}$. Doing the same as previously we get

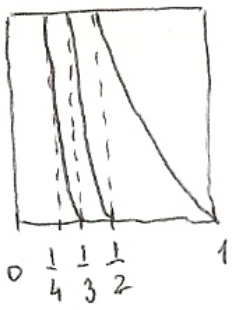
$$\int_X (f + \tfrac{1}{2}) = 0$$
$$\Downarrow$$
$$\int_X d\mu = 0$$ ▨

Examples: The Gauß map

$T(x) = \frac{1}{x} - [\frac{1}{x}] = \{\frac{1}{x}\} \leftarrow$ fraction

$x \in (0, 1]$

$T: [0, 1] \to [0, 1]$



relation to continued fractions (ułamki łańcuchowe)

$0 < x < 1$

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \dots}} = [a_1, a_2, \dots] \qquad a_1, \dots \in \mathbb{N}$$

$$T([a_1, a_2, \dots]) = [a_2, a_3, \dots]$$

Fact: If μ is the absolute continuous with density $\frac{1}{1+x}$

(i.e. $\mu(A) = \int_A \frac{dx}{1+x}$) then μ is T-invariant

It's enough to show $\mu(A) = \mu(T^{-1}(A))$ for A Borel, so

— 〃 — $A = [a,b]$, so

— 〃 — $A = [0,a]$.

So it is enough to check $\int_0^a \frac{dx}{1+x} = \mu([0,a]) \overset{?}{=} \mu(T^{-1}([0,a]))$

$$\ln(a+1) = \sum_{n=1}^{\infty} \int_{I_n} \frac{dx}{1+x} \qquad \left(I_n = \left[\frac{1}{n+a}, \frac{1}{n}\right] \right)$$

$$= \sum_{n=1}^{\infty} \ln\left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+a}} \right) =$$

$$= \cancel{\sum_{n}} \ln\left(\prod_{n=1}^{\infty} \frac{1+\frac{1}{n}}{1+\frac{1}{n+a}} \right)$$

$$= \lim_{N\to\infty} \ln\left(\prod_{k=1}^{N} \frac{1+\frac{1}{k}}{1+\frac{1}{k+a}} \right)$$

$$= \lim_{N\to\infty} \ln\left(\prod_{k=1}^{N} \frac{k+1}{k+1+a} \frac{k+a}{k} \right)$$

$$= \lim_{N\to\infty} \ln\left(\frac{N+1}{N+1+a} \cdot 1+a \right)$$

ln is continuous so we may pass the the limit under the function

$$= \ln(1+a)$$

∎

⌂ Homework

1. Algebraic toral automorphism

Let A integer matrix $n \times n$

$T(x) = Ax$ $\quad T: \mathbb{T}^n \to \mathbb{T}^n$ $\quad \det A = \pm 1$

$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$

Check that T is 1-1 on $\mathbb{T}^n$ and T preserves Leb. measure

2. Let $z \in \mathbb{C}$ $\quad B(z) = \prod_{k=1}^{n} e^{it_k} \frac{z - a_k}{1 - \bar{a}_k z}$ $\quad B(S^1) = S^1$

$a_k \in \mathbb{C}$, $|a_k| < 1$, $t_k \in \mathbb{R}$

Check that B preserves Lebesgue measure if there $\exists_k$ s.t. $a_k = 0$

20.05.2011

$(X, \mathcal{F}, \mu, T)$ $\qquad\qquad$ $\mathcal{G} = \{A \in \mathcal{F} : \mu(A \div T^{-1}(A)) = 0\}$

$\mu(T^{-1}A) = \mu(A)$

$f \in L^1(\mu)$

$$\frac{f + f \circ T + \ldots + f \circ T^{n-1}}{n} \xrightarrow[L^1]{a.e.} E(f|\mathcal{G}) = \bar{f}$$

Def: The map T is ergodic with respect to to μ iff $\mathcal{G}$ is trivial

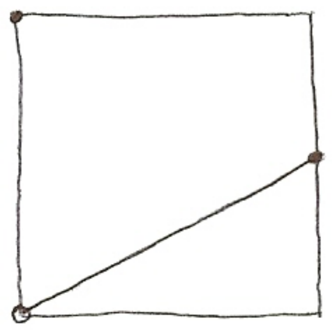
(equivalently : (via ergodic thm) T is ergodic iff $\forall f \in L^1$

$$\frac{f + \ldots + f \circ T^{n-1}}{n} \xrightarrow[L^1]{a.e.} \int f \, d\mu$$

$$E(E(f|\mathcal{G})) = E(f) = \int f \, d\mu$$

Question: $(X, \mathcal{F}, T)$. Does there exist any invariant measure ?

Answer: Example Concider the map $T : [0,1] \circlearrowleft$ $\quad T(x) = \begin{cases} \frac{1}{2}x & x \neq 0 \\ 1 & x = 0 \end{cases}$



Assume that there is an invariant measure μ Then

$$\ldots = \mu((0, \tfrac{1}{4}]) = \mu((0, \tfrac{1}{2}]) = \mu(T^{-1}(0, \tfrac{1}{2}]) = \mu(0, 1]$$

$$\Rightarrow \mu((0,1]) = 0 \qquad \Rightarrow \mu(\{0\}) = 1$$

but it is not invariant as $0 \to 1$

Theorem (Krylov, Bogoliubov): If X is metric compact space and $T : X \to X$ is continuous then there is a Borel, regular T- invariant measure μ.

Proof: Take an arbitrary measure e.g. δ_x

Concider a sequence of measures

$$\mu_n = \frac{\delta_x + \delta_{Tx} + \ldots + \delta_{T^{n-1}x}}{n}$$

then there exists a ~~measure~~ subsequence $\mu_{n_i} \xrightarrow[*]{weakly} \mu$ (some Borel, regular measure)

$$T_* \mu(f) \overset{def}{=} \mu(f \circ T)$$

$$\mu_n = \frac{\delta_x + T_* \delta_x + \ldots + T_*^{n-1} \delta_x}{n}$$

Then $\quad T_*(\mu_{n_i}) = \dfrac{T_*(\delta_x + T_* \delta_x + \ldots + T_*^{n_i - 1} \delta_x)}{n_i} = \dfrac{T_* \delta_x + \ldots + T_*^{n_i} \delta_x}{n_i} =$

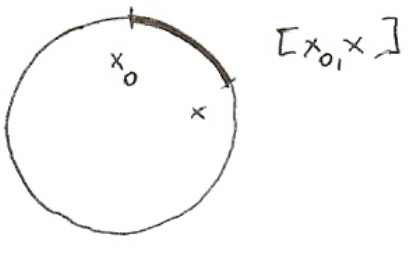
$\swarrow w^*$

$$T_*(\mu) \qquad = \underbrace{\frac{\delta_x + \ldots + T_*^{n_i - 1} \delta_x}{n_i}}_{\searrow \mu} + \underbrace{\frac{T_*^{n_i} \delta_x - \delta_x}{n_i}}_{\searrow 0}$$

$T_* \mu_{n_i}(f) = \mu_{n_i}(f \circ T) \longrightarrow \mu(f \circ T) = T_* \mu$

μ: $T_* \mu = \mu$ $\quad \mu(f \circ T) = \mu(f)$ for every continuous function

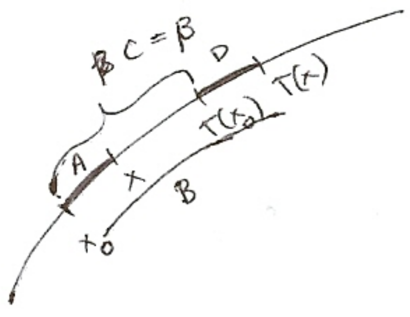
Example: Let $T: S^1 \to S^1$ be homeomorphism of S^1 of degree 1, $\alpha = \rho(T)$ rational. Then (by K-B thm) there exists an invariant measure μ on S^1, μ is non-atomic



Define $h: S^1 \to S^1$

$h(x) = e^{2\pi i \mu[x_0, x]}$

$$\begin{tikzcd}
S^1 \arrow[r, "T"] \arrow[d, "h"'] & S^1 \arrow[d, "h"] \\
S^1 \arrow[r, "z \to e^{2\pi i \beta} z"] & S^1
\end{tikzcd}$$

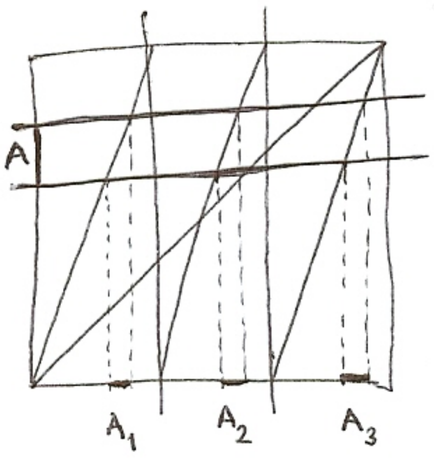


$$h(T(x)) = e^{2\pi i \mu[x_0, Tx]} = e^{2\pi i (C+D)} = e^{2\pi i C} \cdot e^{2\pi i D} = e^{2\pi i \beta} h(x)$$

$\alpha = \beta$

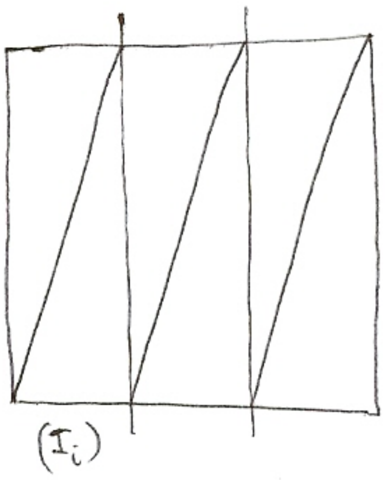
Invariant measures a.c. with respect to Lebesgue measure

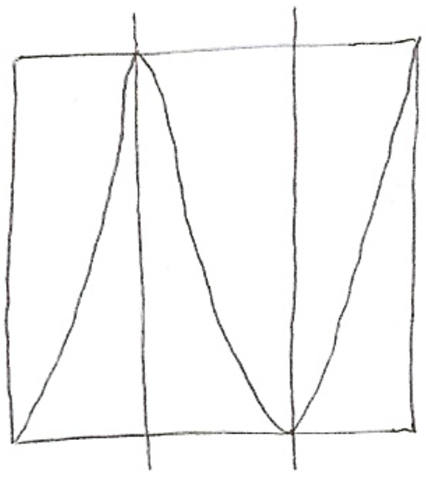
Example



Lebesgue measure is invariant

$|A| = |A_1| + |A_2| + |A_3|)$





$[0,1] = \cup I_i = \cup [a_i, a_{i+1})$ (the last one $[a_j, 1]$)

$f|_{I_i}$ is C^2 on some neighbourhood of I_i

$f|_{I_i} \to [0, 1)$

$f|_{I_i}$ expanding i.e. $|f'| > \lambda > 1$

Assumptions implies that $\log|f'|\Big|_{I_i}$ is Lipschitz continuous with some constant L.

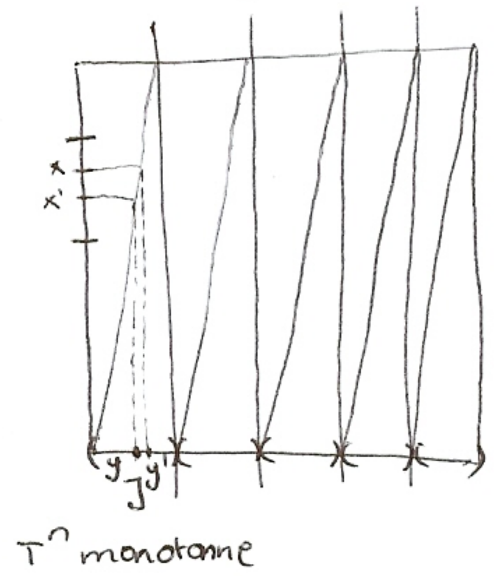
Theorem: This map has exactly one ergodic invariant measure μ a.c. with respect to measure L and

$$C^{-1} < \frac{d\mu}{dL} < C \quad \text{for some} \quad C > 0$$

Proposition: Let $\nu_n = T^n_* L$ $(\nu_n(A) = L(T^{-n}A))$. Then ν_n is a.c. with respect to L with density

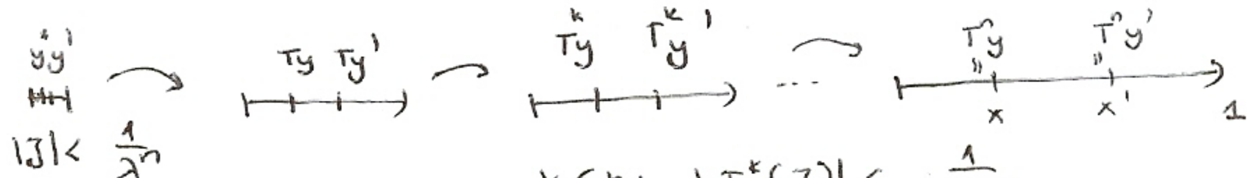
$$g_n(x) = \sum_{y \in T^{-n}(x)} \frac{1}{|(T^n)'(y)|}$$

We shall check that $C^{-1} \leq g_n(x) \leq C$ for all n and some C independent of n.



T^n monotone

$$\begin{tikzcd} J \arrow[r, "T"] \arrow[rrrrr, bend right, "T^n"'] & T(J) \arrow[r, "T"] & T^2(J) \arrow[r] & \dots \arrow[r] & \underbrace{T^{n-1}(J)}_{I_i} \arrow[r, "T"] & {[0,1]} \end{tikzcd}$$



$|J| < \frac{1}{\lambda^n}$

$k \leq n$: $|T^k(J)| \leq \frac{1}{\lambda^{n-k}}$

$$\frac{g_n(x)}{g_n(x')} = \frac{\sum_y \frac{1}{|(T^n)'(y)|}}{\sum_{y'} \frac{1}{|(T^n)'(y')|}}$$

Let us fix a preimage $y \in T^{-n}(x)$ and $y' \in T^{-n}(x')$ paired with y in the way just described.

$$\left|\log \frac{|(T^n)'(y')|}{|(T^n)'(y)|}\right| = \left|\sum_{k=0}^{n-1} \log|T'(T^k(y'))| - \log|T'(T^k(y))|\right| \leq L \sum_{k=0}^{n-1} \frac{1}{\lambda^{n-k}} \leq C$$

We have checked

$$\frac{g_n(x)}{g_n(x')} < C \quad \text{by symetry} \quad C^{-1} < \frac{g_n(x)}{g_n(x')}$$

Now: recall that $\int g_n = 1 \Rightarrow \exists x'$ s.t. $g_n(x') \leq 1 \Rightarrow g_n(x) \leq C$

$$\nu_n = g_n \cdot l$$

$$\mu_n = \frac{\nu_0 + \nu_1 + \dots + \nu_{n-1}}{n}$$

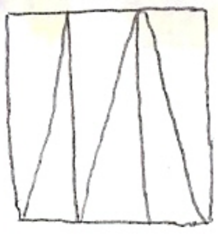
$$\mu_{n_i} = h_{n_i} \cdot l \qquad C^{-1} \leq h_n \leq C$$

$\downarrow$

μ $\quad \forall A \ \mu(\partial A) = 0 \quad \mu_{n_i}(A) \to \mu(A)$

27.05.2011

Ergodicity of invariant measure:



$\log\left|\frac{(f^n)'(x)}{(f^n)'(y)}\right| < C$

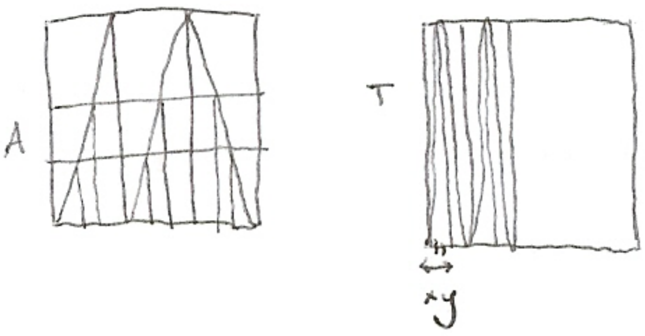
Assume $A \subset I$ is a Borel invariant set

$T^{-1}(A) = A$

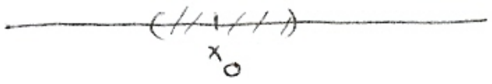
$0 < \mu(A) < 1$

$\Downarrow$

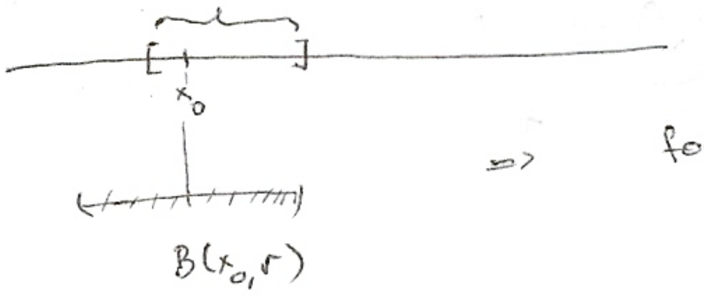
$0 < Leb(A) < 1$



Take a density point of A (x_0)



$I_n(x_0)$ - piece of monotonicity of T^n, containing x_0

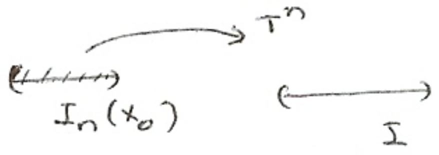


$\Rightarrow$ for large n $\quad \frac{Leb(I_n(x_0)\cap A)}{Leb(I_n(x_0))} \swarrow 1$

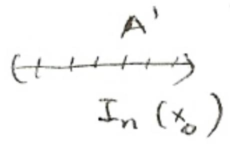
$$\frac{Leb(B(x_0,r)\cap A)}{Leb(B(x_0,r))} \xrightarrow[r\to 0]{} 1$$

$$Leb(A'\cap I_n(x_0)) = Leb(T^{-n}A'\cap I_n(x_0)) \geq \inf \frac{1}{|(T^n)'|}_{I_n(x_0)} \cdot Leb(A')$$

$$Leb(I_n(x_0)) \leq \sup \frac{1}{|(T^n)'|}\Big|_{I_n(x_0)}$$



$$\frac{Leb(A'\cap I_n(x_0))}{Leb(I_n(x_0))} \geq \underbrace{\inf\left|\frac{(T^n)'(x)}{(T^n)'(y)}\right|\Big|_{I_n(x_0)}}_{> c > 0}$$



Hyperbolic sets - Ergodicity

Example: $T: \mathbb{T}^2 \to \mathbb{T}^2$ hyperbolic isomorphism of $\mathbb{T}^2$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Proposition: The Lebesgue measure $Leb|_{\mathbb{T}^2}$ (invariant for T) is ergodic.
(This argument is very general!)

Proof: We want to show that for every $h \in L^1(Leb)$

$$(*)\ \frac{1}{n}\sum_{i=10}^{n-1} h\circ T^i \longrightarrow \int h\, dleb$$

~~$(*)\ \frac{1}{n}\sum_{i=0}^{n-1} h\circ T^i \longrightarrow$~~

① we prove $(*)$ for $h = g$ - continuous.

we know from Birkhoff ergodicity theorem that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g \circ T^{i} = \mathbb{E}(g \mid G_+) \qquad G_+ = \{A : \mathrm{Leb}(T^{-1}(A) \div A) = 0\}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g \circ T^{-i} = \mathbb{E}(g \mid G_-) \qquad G_- = \{A : \mathrm{Leb}(T(A) \div A) = 0\}$$

$\downarrow$ from $(**)$

$\int g \, d\mathrm{leb}$

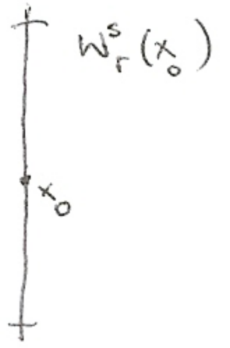
$G_+ = G_-$

<u>Theorem:</u> for a.e. $x_0 \in \mathbb{T}^2$

$$\{x_0 : \lim \frac{1}{n} \sum_{i=0}^{n-1} g \circ T^{i} = \lim \frac{1}{n} \sum_{i=0}^{n-1} g \circ T^{-i}\}$$

$G =$

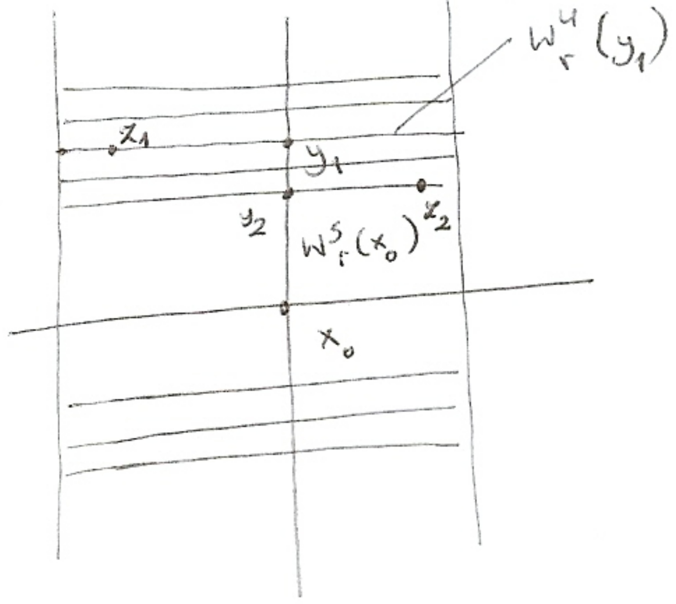
$\mathrm{Leb}(G) = 1$



There exist a set $\mathcal{F}$ of full measure in $\mathbb{T}^2$ s.t for every $x_0 \in \mathcal{F}$:

$\mathrm{Leb}_1(W^s_r(x_0) \cap G) =$ full measure in $W^s_r(x_0)$

$y \in W_r^s(x_0) \cap G$



We have checked that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g \circ T^{-i}(z_1) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g \circ T^{-i}(z_2) \qquad (**)$$

It remains to prove the statement for an arbitrary $h \in L^1$

$$C(\mathbb{T}^2) \ni g_n \xrightarrow{L_1} h \qquad \int |g_n - h| \, d\mathrm{leb} \to 0$$

$g \in C(\mathbb{T}^2)$

$$h_+ = \lim_{n\to\infty} \frac{1}{n} \sum h \circ T^{i} \qquad = \int |g - h| \, d\mathrm{leb} < \varepsilon$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum g \circ T^{i} + \lim_{n\to\infty} \frac{1}{n} \sum (h-g) \circ T^{i}$$

$\downarrow$ we already know

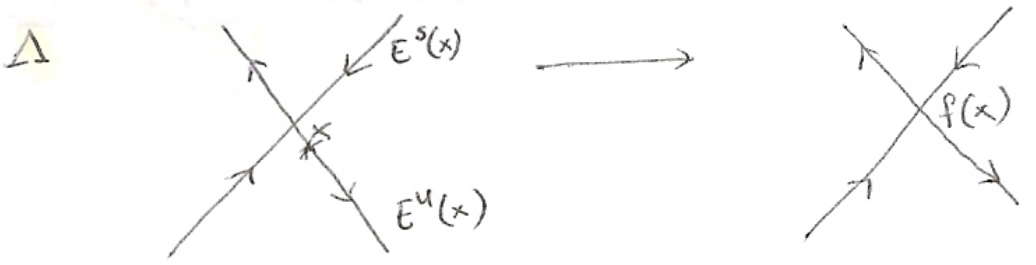
$\int g \, d\mathrm{leb}$

preserves measure

$$(h_+ - \int g) = \lim \frac{1}{n} \sum_{i=0}^{n-1} (h-g) \circ T^{i} = \| h - g \|_{L^1} < \varepsilon$$

Thus h_+ differs (in L^1) from a constant $\int g$ by less ten ε.

Hyperbolic sets:



Proposition: Let Λ be a hyperbolic set. Then $E^u(x)$, $E^s(x)$ depend continuously on x.

---

x_n $\quad v_1^n \dots v_s^n$ - orthonormal basis in $E^s(x_n)$

$\downarrow \quad \downarrow$

it may be a sub-sequence $\quad v_1^0 \quad v_s^0$ $\quad$ We claim that $v_1^0, \dots, v_s^0 \in E^s_x$.

$v_1^n \in E^s(x_n)$ $\quad \lambda > 1$

$$\|D^k_{x_n} f(v_1^n)\| < \lambda^{-k} \|v_1^n\|$$

$$\Big\downarrow n \to \infty$$

$$\|D^k_x f(v_1^0)\| < \lambda^{-k} \|v_1^0\|$$

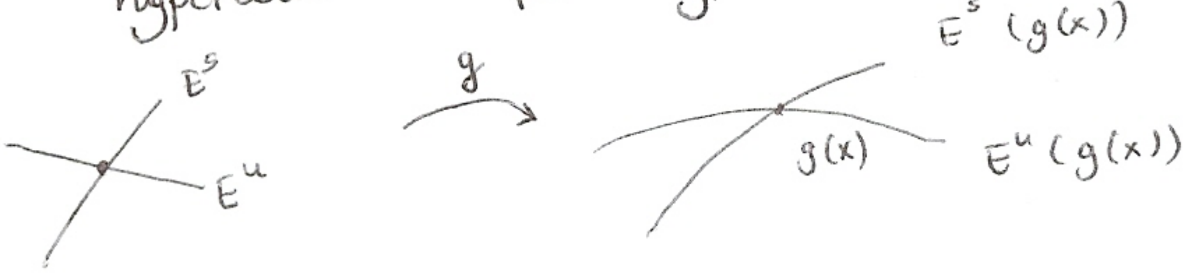
3.06.2011

Stability of hyperbolic sets:

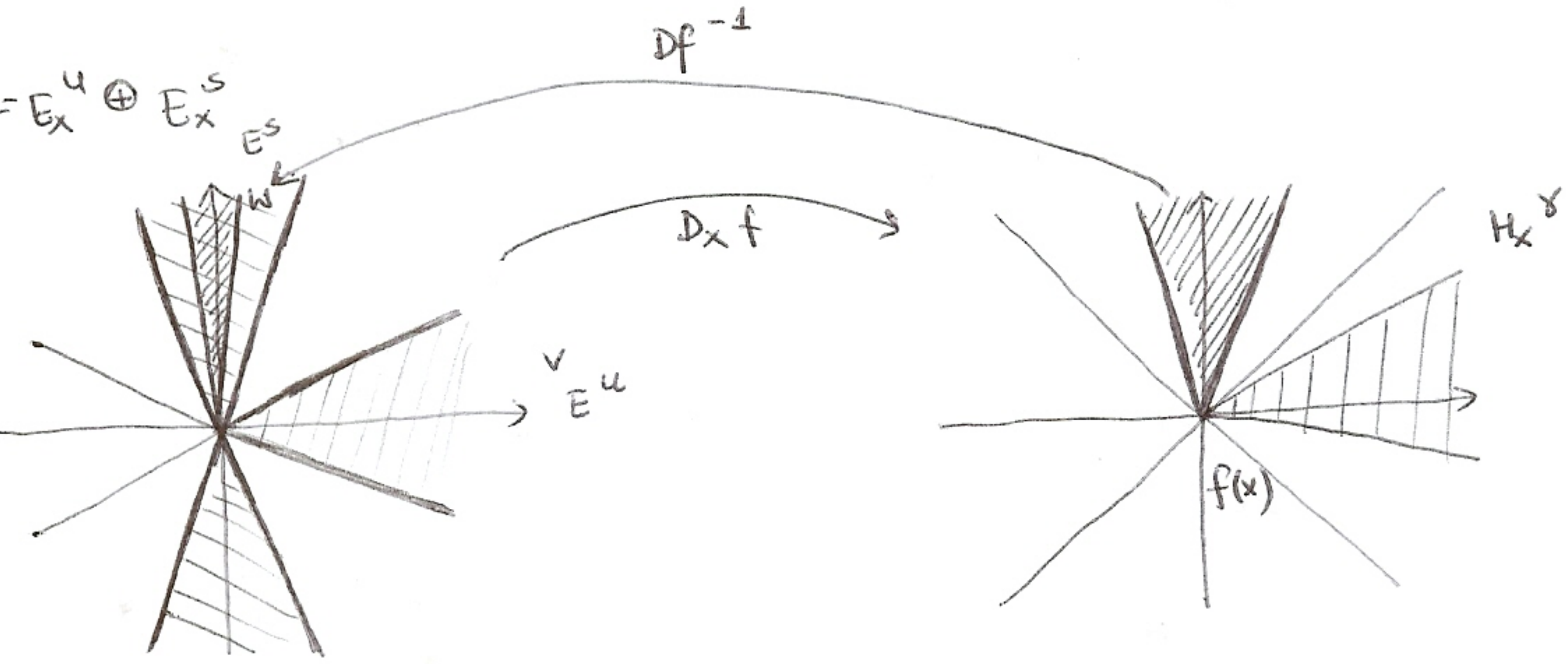
Proposition: Let Λ be a hyperbolic set for $f: M \to M$ and let $g: M \to M$ be sufficiently close to f (in C^1 metric). Then there exists a neighbourhood $V \supset \Lambda$ in M such that the set

$$\Lambda^g_V = \{ x : \forall_n \ g^n(x) \in V \}$$ ~~jest zbiorem hiperbolicznym~~

is hyperbolic for g.



Proof:
$T_xM = E_x^u \oplus E_x^s$



$$H_x^\gamma = \{ (\underset{\substack{\cap \\ E^u}}{v} + \underset{\substack{\cap \\ E^s}}{w}) : \|w\| \le \gamma \|v\| \}$$

$V_x^\gamma = \{ (v+w) : \|v\| \le \gamma \|w\| \}$

$\|D_x f^n(w)\| \le C \|w\| \lambda^{-n}$

$\|D_x f^n(v)\| \ge \|v\| \lambda^n$

(*) $D_x f(H_x^\gamma) \subset H^\gamma_{f(x)} \lambda^{-2}$

we have a splitting for Λ. we already know that E_x^s, E_x^u depend continuously on $x \in \Lambda$. we extend the spliting E_x^s, E_x^u continuously to some neighbourhood of Λ.

we check that (*) holds not only in Λ but also in some neighbourhood of Λ.

Lemma: Let $L_m : \mathbb{R}^u \oplus \mathbb{R}^s \to \mathbb{R}^u \oplus \mathbb{R}^s$ be a sequence of linear maps $\lambda > 1$

(1) $L_m(H^\gamma) \subset \mathrm{Int}\, H^\gamma$

(2) $L_m^{-1}(V^\gamma) \subset$ ~~tam~~ $\mathrm{Int}\, V^\gamma$

(3) $\|L_m(u)\| \ge \lambda \|u\|$ for $u \in H^\gamma$

(4) $\|L_m(u)\| \le \lambda^{-1} \|u\|$ for $u \in V^\gamma$

Then $E_m^u = \bigcap_{i=0}^{\infty} L_{m-1} \circ L_{m-2} \circ \dots \circ L_{m-i} H^\gamma$

is a family of invariant subspaces: $L_m : E_m^u \to E_{m+1}^u$

Invariant spliting

$E_m^u \oplus E_m^s$ $\quad E_m^s = \bigcap_{i=0}^{\infty} L_m^{-1} \circ L_{m+1}^{-1} \circ \dots \circ L_{m+i}^{-1} V^\gamma$

$L_m : E_m^s \to E_{m+1}^s$ ← contracting

Def: $f : M \circlearrowleft$ diffeomorphism is called Anosov diffeomorphism if the whole manifold M is an hyperbolic set.

Theorem: If $f: M \to M$ is a C^1 Anosov diffeomorphism then f is structurally stable

Sketch: $g \sim f$ C^1- close

Then $g : M \to M$ is hyperbolic (follows from proposition)

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "h"'] & M \arrow[d, "h"] \\ M \arrow[r, "g"] & M \end{tikzcd}$$

?

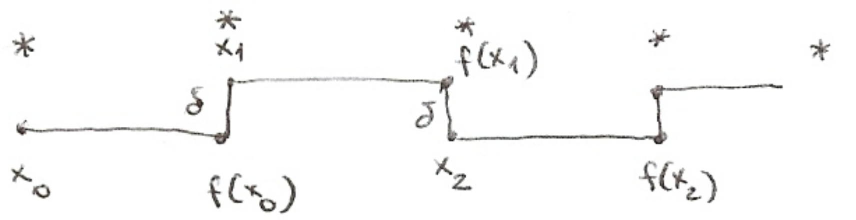
Shadowing lemma (lemat o ε trajektoriach):

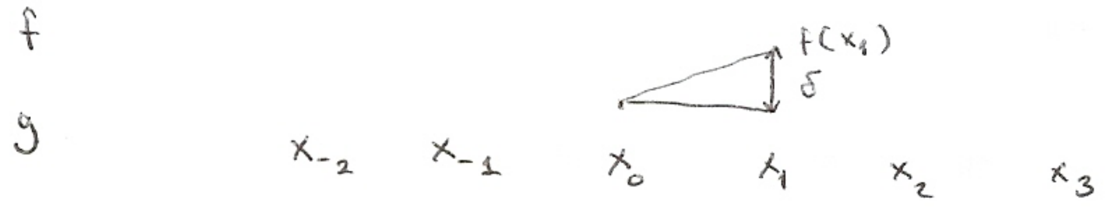
Def: A sequence $x_0\, x_1\, x_2 \dots x_n \dots$ is an ε trajectory if $d(x_{i+1}, f(x_i)) < \varepsilon$.

Proposition: If f is an Anosov diffeomorphism then

(**) $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ such that every δ-trajectory can "be ε-shadowed" by a "real" trajectory. i.e.

$\exists\, y \in M$ such that $d(f^i y, x_i) < \varepsilon$.





trajectory of g is a δ-trajectory for f.

$h(x) =$ a unique y defined by (* *)

Continuity of h. Suppose not. Then there exists a sequence $x_k \to x$ s.t. $d(h(x_k), h(x)) \geq \delta_0 > 0$

Assume that $h(x_k) \longrightarrow z \in M$

$$d(f^i h(x), f^i(z)) \leq d(f^i h(x), g^i(x)) + d(g^i x, g^i(x_k)) + \underset{\substack{\downarrow k\to\infty \\ 0}}{d(g^i x_k, f^i h(x_k))} +$$

$$+ \underset{\substack{\downarrow k\to\infty \\ 0}}{d(f^i h(x_k), f^i(z))} < 4\varepsilon$$

Conclusion: For all $i \in \mathbb{Z}$ $d(f^i h(x), f^i(z)) < 4\varepsilon$

$$\Downarrow$$

$$h(x) = z$$