APPROXIMATION PROPERTY IN B SPACES :

Def: A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is called Schauder basis if

$\forall x \in X$. $x = \sum_{n=1}^{\infty} a_n^{(x)} x_n$ uniquely . Then

$P_n = \sum_{k=1}^{n} a_k(x) x_k$ ← bounded projections , $P_n \to Id_X$

By S-S , there is a uniform bound on P_n's $\|P_n\| \le k < \infty$

l_p, c_0, $L_p(\mathbb{T})$, $C(I)$

⇒ X - separable

Does ~~a~~ every separable Banach space have a basis ?

Notation:

$L(X,Y)$ - all linear, bounded operators from X to Y

$\cup$

$F(X,Y)$ - finite rank

$K(X,Y)$ = compact operators

T compact $\iff \overline{TB_X}$ - compact

Obs:

$\overline{F(X,Y)} \subset K(X,Y)$

"⊃" needs more properties of X or Y (we'll come back to it)

Def: $X \in BAP$ (Bounded app. property) if $\exists \lambda$ $\forall D \subset X$, finite $\forall \varepsilon > 0$

$\exists T \in F(X)$ so that $\|Tx - x\| < \varepsilon$ $\forall x \in D$, $\|T\| < \lambda$.

easy : X separable , $X \in BAP \iff \exists T_n \in F(X)$, $T_n \to Id_X$ strongly (in every point)

Immediate : X has a basis ⇒ X has BAP

Example : $L_p(\mu) \ni f_1, \dots, f_n$ ε . Let $g_1, \dots, g_n$ be simple functions so that $\|f_j - g_j\| < \varepsilon$

$g_j = \sum_{i=1}^{n_j} a_{ji} \mathbb{1}_{A_{ji}}$, let $\mathcal{A}$ = the algebra generated by $\{A_{ji}\}$

Let $Tf = \mathbb{E}(f|\mathcal{A})$

Example 2: $X = L_p(\mathbb{T})$ or $C(\mathbb{T})$, Y - translation invariant $\subset X$

$1 \le p \le \infty$ i.e. $Y = X_A = \{e^{ikt} : k \in A\}$

Let $\varphi_n f = \sum_{k=-n}^{n} \hat{f}(k) e^{ikt}$; $T_n = \frac{1}{n+1}(P_0 + P_1 + \dots + P_n)$

By Fejer's theorem , $\|T_n\| = 1$, $T_n f \to f$ in $L_p(\mathbb{T})$, $p < \infty$
$C(\mathbb{T})$

Notice that $P_n X_A \in X_A$ and $T_n X_A \subset X_A$

$\Rightarrow$ ~~A~~ X_A has BAP (except $p = \infty$)

open question: 1° $L_\infty(\mathbb{T})_N = H_\infty \in$ BAP ?

$C(\mathbb{T})_A$ - this is known ($\in$ BAP)

2° when does it have basis ?

By Bočkariov '74, $C(\mathbb{T})_N$ = disc algebra does have a basis.

$L_p(\mathbb{T})_{\mathbb{N}}$ isomorphic to $L_p(\mathbb{T})$

$\|$ (from Banach space point of view)

H_p

$H_1 \not\simeq L_1$

Example 3: $C^*(G) \in$ BAP for some G's, e.g. Haagerup proved it for $G = F_2$.

Def: $X \in AP$ (Grothendick's AP) if $\forall K \subset X$ compact $\forall \varepsilon > 0$ $\exists\, T \in F(X)$ so that $\|Tx - x\| < \varepsilon$ $\forall x \in K$.

(there is no lambda)

Easy: BAP $\Rightarrow$ AP

K ε

$\cup$

D - finite, $\varepsilon/3\lambda$

there is $T \in F(X)$, $\|Tx - x\| < \varepsilon/3$ $\Leftarrow$

$\|T\| \le \lambda$, $\forall x \in D$,

MAIN THEOREM (Enflo 72) there is separable B space without AP

(T-F tu był taki napis)

thm: AP $\not\Rightarrow$ BAP $\not\Rightarrow$ basis Szarek

WHY AP ?

thm 1: $X \in AP \iff \forall Y,\ K(Y,X) = \overline{F(Y,X)}$

Open question: Is it true that $X \in AP$ iff $K(X) = \overline{F(X)}$

Proof: "$\Rightarrow$" $T \in K(Y,X)$, let $K = \overline{TB_Y}$, let $S: F(X)$ be such that $\|Sx - x\| < \varepsilon$ $\forall x \in X$. Then for $T_1 = ST$ we have

$\|T_1 x - x\| < \varepsilon$ $\forall y \in B_Y$

zawsze wyst. w parach można normować

Thm 2: $X \notin AP \iff \exists\, x_1^*, x_2^*, \ldots \in X^*$, $x_1, x_2, \ldots \in X$ so that

$\sum \|x_n^*\|\|x_n\| < \infty$ (1), $\sum x_n^*(x_n) = 1$ (2), $\sum x_n^*(x)x_n = 0$ (3), for every $x \in X$

$0 = x_j^*\left(\sum x_n^*(x_k)x_n\right) = \sum_{n=1}^{\infty} x_n^*(x_k)x_j^*(x_n) = 0$ $\forall j,k$

Denote $a_{nk} = x_n^*(x_k)$ $\quad \sum_{n=1}^{\infty} a_{nk}a_{jn} = 0 \iff A^2 = 0$ $\quad A = (a_{nk})_{n,k=1}^{\infty}$

(3') $\rightarrow$

$A = \cancel{(a_{nk})}_{n,k=1}$ $\quad tr\, A = \Sigma a_{nn} = 1 \quad (2)$

$\Uparrow \quad \|x_n^*\| \to 0 \quad , \quad \Sigma \|x_n\| < \infty \quad (11)$

$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = 0 \quad (1')$

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Thm (Grothendick) = thm 2

Lemma (1): $K \subset X$ compact iff $\exists\, x_n \to 0$ s.t. $K \subset \overline{conv}\, \{x_n\}_{n=1}^{\infty}$

proof: $\Uparrow$

In $L(X)$ introduce the loc. convex topology τ given by the seminorms

$\|T\|_K = \max \{\|Tx\| : x \in K\}$, K-compact, $K \subset X$

Lemma 2: $(L(X), \tau)^* \ni \Lambda$ iff $\exists\, x_1, x_2, \dots, x_1^*, x_2^*, \dots$ s.t. (1)

$\Lambda T = \sum_{n=1}^{\infty} x_n^* (T x_n)$

Proof: $L^* \ni \Lambda$ iff $\exists\, K \subset X$ compact so that $|\Lambda T| \le \|T\|_K$ for every $T \in L$

by lemma 1 we can assume that $K = \{x_n\}_{n=1}^{\infty}$, $x_n \to 0$

(wlg)

Let $Y \subset c_0(X) = (X \oplus X \dots)_{c_0} \ni (Y_1, \dots, Y) \quad \|\cdot\|_{c_0} = \max \|Y_n\|$

$\|$

$\{\{Tx_n\}_{n=1}^{\infty} : T \in L\}$

Let us define for $y \in Y$, $y = (Tx_1, Tx_2, \dots)$, $\Phi y = \Lambda T$

$y = (Sx_1, Sx_2, \dots)$ $\quad \Phi y = \Lambda S$ "?"

$\Lambda(S-T) \overset{?}{=} 0$, $((S-T)x_1, (S-T)x_2, \dots) = (0, \dots)$

$|\Lambda(S-T)| \le \|S-T\|_K = 0$

$|\Phi y| \le \|y\|_{c_0}$ - continuous

By Hahn-Banach there is $\psi \in c_0(X)^*$ so that $\Phi y = \psi_y$ for $y \in Y$

$\|$

$\ell_1(X^*)$, i.e. $\psi(y_1, y_2, \dots) = \sum_{n=1}^{\infty} x_n^*(y_n)$

In particular, $\Lambda T = \Phi(Tx_1, Tx_2, \dots) = \psi(Tx_1, Tx_2, \dots) = \sum_{n=1}^{\infty} x_n^*(Tx_n)$

Proof of thm: A version of H.B

L is locally convex, $M \subset L$ a (linear) subspace, $x_0 \notin \overline{M}$ iff

$\exists\, \Lambda \in L^*$ s.t. $\Lambda_{|M} = 0$, $\Lambda x_0 = 1$

take $L = (L(X), \tau)$, $M = F(X)$: $X \notin AP \iff Id_X \notin \overline{F(X)}^{\tau}$

$\exists\, \Lambda \in (L(X), \tau)^*$ so that $\Lambda_{|M} = 0$, $\Lambda(Id) = 1$, $\Lambda T = \Sigma x_n^*(Tx_n)$

$\Lambda(Id) = \Sigma x_n^*(x_n) \quad (2)$ $\qquad \Updownarrow$

Notation : $(x^* \otimes x)(y) = x^*(y) x$ $\qquad \Lambda(x^* \otimes x) = 0 \quad \forall x^*, x$

$\Lambda(x^* \otimes X) = \quad x^*\left(\sum x_n^*(x) x_n\right) = 0 \iff \sum x_n^*(x) x_n = 0 \quad \forall x$ (3)

Remark: $X^* \in AP \;\not\Rightarrow\; X \in AP$

$X \notin AP \Rightarrow X^* \notin AP \iff \exists\, x_n^*, \underset{\substack{\cap \\ X^{**}}}{x_n^{**}}$

Ex:

$\underset{\substack{| \\ \text{nuclear} \\ \text{operators}}}{N(H)^*} = \underset{\substack{| \\ \text{bounded} \\ \text{operators}}}{L(H)}$

$L(H) \notin AP$

$N(H) \in AP$

Tensor norms: $Y \otimes X$, $\sum_{i=1}^{k} y_i \otimes x_i \sim \sum_{i=1}^{n} u_i \otimes v_i$

$\sum y^*(y_i) x^*(x_i) = \sum y^*(u_i) x^*(v_i)$

$\sum_{i=1}^{n} y_i \otimes x_i = \underset{\{-1,1\}^n = \mathcal{E}}{Av}\left[\left(\sum \xi_j y_j\right) \otimes \left(\sum \varepsilon_k x_k\right)\right]$

$\qquad = 2^{-n} \sum_{\varepsilon}$

$\beta \in Y \otimes X$

Proj. norm: $\|\beta\|_\wedge = \inf\left\{\sum_{i=1}^{k} \|y_i\|\|x_i\| : \beta = \sum y_i \otimes x_i\right\}$

$Y = X = \ell_p$, $y_i = x_i = e_i$

$k^{1/p} = \left\|\sum_{j=1}^{k} \varepsilon_j x_j\right\| = \|y\|$

Inj. norm:

$\|\beta\|_\vee = \sup_{\substack{1 = \|x^*\|, \\ 1 = \|y^*\|}} \left|\sum y^*(y_i) x^*(x_i)\right|$

$Y = X^*$, $\beta = \sum_{n=1}^{k} x_n^* \otimes x_n$; $\beta \in X^* \otimes X \longleftrightarrow F(X)$

$\beta(x) = \sum x_n^*(x) x_n$

$\|\beta\|_\vee = \sup\{\|\beta(x)\| : \|x\| \le 1\}$

Obs: $\|\beta\|_\wedge \ge |tr\,\beta| \overset{def}{=} \left|\sum x_n^*(x_n)\right|$

$\|\beta\|_\vee \le \|\beta\|_\wedge$

Define: $X^* \hat{\otimes} X$, $\underset{\substack{\| \\ F(X)}}{X^* \check{\otimes} X}$ completions of $X^* \otimes X$ in the ~~appr~~ proj., inj. norms.

(1)

Fact: $X^* \hat{\otimes} X = \left\{\sum_{n=1}^{\infty} x_n^* \otimes x_n : \sum \|x_n^*\|\|x_n\| < \infty\right\} / \sim$

where „$\sim$":

$$\sum_{n=1}^{\infty} x_n^* \otimes x_n \sim \sum_{n=1}^{\infty} u_n^* \otimes v_n \quad \text{iff} \quad \Big(\sum_{n=1}^{k} x_n^* \otimes x_n - \sum_{n=1}^{k} u_n^* \otimes v_n \Big)_\wedge \xrightarrow{k\to\infty} 0$$

<u>define:</u> $\operatorname{tr}\left(\sum_{n=1}^{\infty} x_n^* \otimes x_n\right) = \sum x_n^*(x_n)$

(1)   (2) $\operatorname{tr}\beta = 1$ (3) $\beta(x) = 0 \quad \forall x$

$$X^* \hat{\otimes} X \longrightarrow X^* \check{\otimes} X$$

noninjective map

$\Updownarrow$

$X \notin AP$

<u>23.03.2011</u>

$X \notin AP \iff \exists\, \beta = \sum_{k=1}^{\infty} x_k^* \otimes x_k \in X^* \hat{\otimes} X, \quad \sum_1^{\infty} \|x_k^*\| \|x_k\| < \infty$ (1)

$\operatorname{tr}\beta = \sum_1^{\infty} x_n^*(x_n) = 1$, (2) $\qquad \beta(x) = 0 \quad \forall x \in X$

$\qquad\qquad \beta(x) = \sum x_n^*(x) x_n$

<u>Enflo approach:</u>

$X \notin AP \iff \exists\, \beta_n \in X^* \otimes X;$

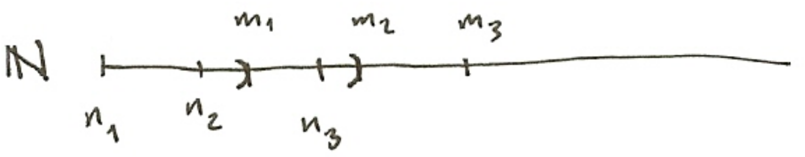
(1) $\sum \|\beta_{n+1} - \beta_n\|_\wedge < \infty$

(2) $\operatorname{tr}\beta_n = 1 \quad \forall n$

(3) $\|\beta_n\|_\vee \xrightarrow{n\to\infty} 0$

$$\beta = \beta_1 + \sum_{n=1}^{\infty} \beta_{n+1} - \beta_n$$

<u>Thm:</u> A <u>random</u> $\frac{\infty}{2}$ dim subspace of ℓ_p, $p > 2$ fails AP



$[n_j, m_j]$ they can not be disjoint

$n_j < m_{j-1}$

$X = (a_1, a_2, a_3, \dots)$: $a_j \in \mathbb{R}^{3\cdot 2^j}$, $j = 0, 1, 2, \dots$

$\underbrace{a_1}_{3}, \underbrace{a_2}_{6}, \underbrace{a_3}_{12}$

$\| \; \mathbb{R}^\infty$

$$T^k \in \mathcal{O}(3\cdot 2^k) \longrightarrow \left|\begin{array}{c} U_k \\ \hline V_k \end{array}\right| \begin{array}{l} 2^k \\ 2^{k+1} \end{array} \Big\} \; 3\cdot 2^k$$

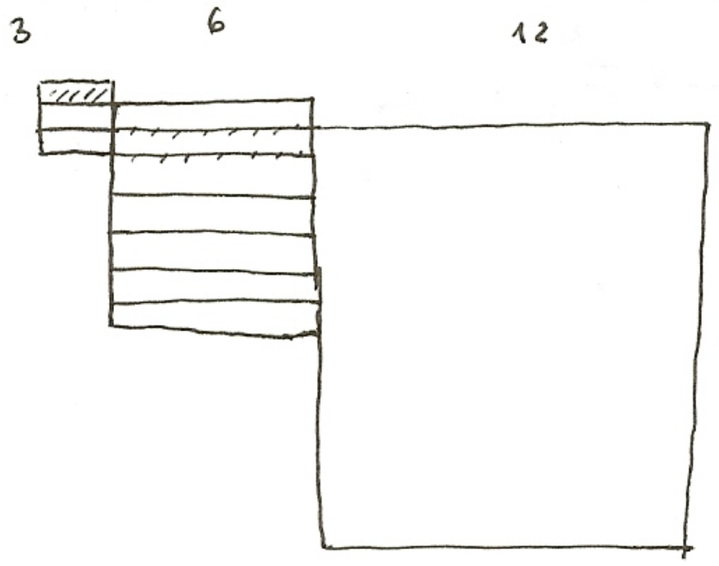
$u_1^k, \dots, u_{2^k}^k$ - rows of U_k

$v_1^k, \dots, v_{2^k}^k$ - rows of V_k

$\cap \; \mathbb{R}^{3\cdot 2^k}$

$\mathcal{O}(n)$ = orthogonal group on $\mathbb{R}^n$

$$y_i^k \overset{\text{def}}{=} v_i^{k-1} + u_i^k$$



$Y = Y(\{T_k\}) = \operatorname{span}\{y_i^k : k = 0, 1, \dots, i = 1, \dots, 2^k\}$

Y_p = completion of Y in L_p

Thm: $\mathbb{P}(Y(T_k) \not\in AP) = 1$, $\bigotimes_{k=1}^{\infty} \mathcal{O}(3\cdot 2^k)$

$\beta_n = 2^{-n} \sum_{i=1}^{2^n} \varphi_i^n \otimes y_i^n$ where $\varphi_i^k(x) = \langle u_i^k, x\rangle$

depends on T_{n-1}, T_n — for $x \in Y$ — $= \langle v_i^{k-1}, x\rangle$

enough to verify for $x = y_j^m$

$\beta_n(x) = 2^{-n}$ orthogonal projection on span $y_1^n, \dots, y_{2^n}^n$

$\|\beta_n\|_V \sim 2^{-n} \to 0$

Thm: $\mathbb{P}((T_{n-1}, T_n, T_{n+1}) : \|\beta_{n+1} - \beta_n\| < Cn 2^{-n}) \geq 1 - 2^{-n}$

Thm: $\mathbb{P}(Y_p(T) \not\in Ap) = 1$ (this is implied by the previous thm)

$\beta_n = 2^{-n} \sum_{i=1}^{2^n} \varphi_i^n \otimes y_i^n = 2^{-n}\left(\sum \varphi_i^n \otimes v_i^{n-1} + \sum \varphi_i^n \otimes u_i^n\right) =$

depends on T_{n-1}, T_n

$= 2^{-n}\left(\sum^{2^n} u_i^n \otimes v_i^{n-1} + \sum^{2^n} u_i^n \otimes u_i^n\right) =$

$\beta_{n+1} = 2^{-n-1}\left(\sum^{2^{n+1}} v_i^n \otimes v_i^n + \sum^{2^{n+1}} v_i^n \otimes u_i^{n+1}\right.$

Pure Ist year algebra:

$\sum_{i=1}^{N} x_i \otimes y_i$, $x_i = \sum a_{ij} u_j$, $y_i = \sum b_{ik} v_k$ $A = (a_{ij})_{i=1}^N$, $B = (b_{ik})_{i=1}^N$

$\|$

$\sum_i u_j \otimes d_j$

$[\; A \;]\,[\; B \;]$

$d_j = \sum A^* B v_k$

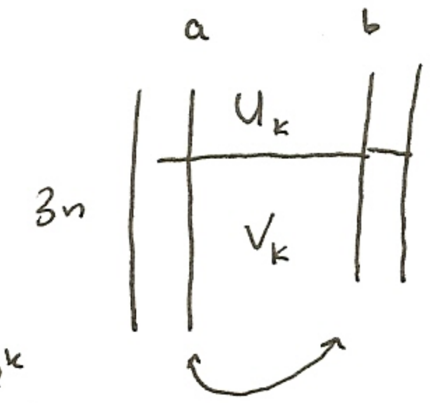
- - - -

$2^{-k-1} \sum v_i^k \otimes v_i^k - 2^{-k} \sum u_i^k \otimes u_i^k = 2^{-k} \sum e_i^k \otimes V_k V_k^* e_i^k$

$\mathbb{R}^{3\cdot 2^k}$ — $\mathbb{R}^{3\cdot 2^k}$ — u.v.b of $\mathbb{R}^{3\cdot 2^k}$

$$\underbrace{(2^{-k} V_k V_k^* - \underset{2^{-k-1}}{U_k U_k^*})\, e_i^k}_{d_i^k}$$

$\|\beta_n - \beta_{n+1}\|_\wedge \leq 2^{-k} \max \|d_i^k\|_p$



random orth. matrix (?)

<u>Fact:</u> with probability $> 1-2^{-k}$

$|\langle a, b\rangle| < C\cdot k^2 \cdot 2^{-k/2}$

Conclusion:

$\|d_i^k\|_\infty \leq C\cdot k^2 \cdot 2^{-\frac{k}{2}}$, $\|d_i^k\|_p \leq C\cdot k^2\cdot 2^{-\frac{k}{2}} \|\underbrace{(1,\dots,1)}^{3\cdot 2^k}\|_p \leq C\cdot k^2 \cdot 2^{-k/2 + k/p}$

$\|\beta_n - \beta_{n+1}\|_1 \leq C\cdot k^2 \cdot 2^{k(1/p - 1/2)}$

the consts. does not work with $p \leq 2$

$Y_p \sim \ell_2$ for $1 < p < 2$

almost surely