$\mu \in M(\mathbb{T})$ is idenpotent if $\mu * \mu = \mu$ (WOJCIECHOWSKI)

Are we able to charakct. the seq. of ~~...~~ ...

<u>Thm:</u> suppose that $\|\mu\| < M$ $\quad \mu * \mu = Id$

then $\hat{\mu} = \sum_{i=1}^{L} \mathbb{1}_{x_i + H_i}$ $\quad$ 1) $L < e^{e^{cM^4}}$

$\uparrow$ sub-groups

2) # diffrent subgroups $\leq M + \frac{1}{100}$

$f : G \to \mathbb{C}$, $\quad \|f\|_p = \left(\frac{1}{|G|} \sum_{x \in G} |f(x)|^p \right)^{1/p}$

$g : \hat{G}$ $\quad \|g\|_p = \left(\sum_{\gamma \in \hat{G}} |g(\gamma)|^p \right)^{1/p}$

$\hat{f}(\gamma) = \mathbb{E} f(x) \overline{\gamma(x)}$

$f_1 * f_2(t) = \mathbb{E}_{x \in G} f_1(x) f_2(t - x)$

$\|f\|_A = \|\hat{f}\|_1 = \sum_{\gamma \in \hat{G}} |\hat{f}(\gamma)|$

1. step

suppose $f : G \to \mathbb{R}$, $\quad \|f\|_A \leq M \geq 1$

$d(f, \mathbb{Z}) \leq e^{-C_1 M^4}$

$\quad ||$

$\sup_{x \in G} d(f(x), \mathbb{Z})$ $\qquad f_{\mathbb{Z}}(x) :=$ closest integer to $f(x)$

$f = f_1 + f_2$

(1) $\|f_1\|_A \leq \|f\|_A - \frac{1}{2}$ $\quad$ or else

$(f_1)_{\mathbb{Z}} = \sum_{j=1}^{L} \pm \mathbb{1}_{x_i + H_i}$ $\quad$ H- subgroup $\quad L < e^{e^{c' c_0 M^4}}$

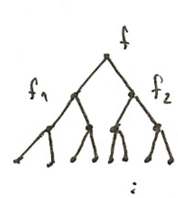
(2) $\|f_2\|_A \leq \|f\|_A - \frac{1}{2}$ $\quad d(f_1, \mathbb{Z}) \leq d(f, \mathbb{Z}) + \varepsilon$

$d(f_2, \mathbb{Z}) \leq 2d(f, \mathbb{Z}) + \varepsilon$

$f : G \to \mathbb{Z}$, $\quad \varepsilon = e^{-C_0 M^4}$ $\quad C_0 >>> C_1$

$f = f_1 + f_2$ $\quad d(f_1, \mathbb{Z}), d(f_2, \mathbb{Z}) \leq \varepsilon$

$(f_i)_{\mathbb{Z}}$ is a sum of $e^{e^{cM^4}}$ functions $\pm \mathbb{1}_{x_i + H}$



2M+1

$\|f_j\|_A \leq \|f\|_A - \frac{1}{2}$ $\quad i = 1, 2$ $\quad$ and so on

$$f = \sum_{k=1}^{L} f_k \quad \begin{cases} L & \leq 2^{2M-1} e^{e^{CM^4}} \\ \forall k & (f_k)_{\mathbb{Z}} = \sum \pm \mathbb{1}_{x_{k,j}} + H_k \\ d(f_k, \mathbb{Z}) & \leq 2^{2M} \cdot \varepsilon \end{cases}$$

$$\| f - \sum_{k=1}^{L} (f_k)_{\mathbb{Z}} \|_\infty \leq 2^{4M-1} \cdot \varepsilon < 1$$

$$\|f\|_A = \sum^{L} \|f_k\|_A$$ (we skip 2) of thm) for now (?)

Bourgain System

G - finite abelian group

$d \geq 1$, $d \in \mathbb{Z}$

$S = (X_\rho)_{\rho \in [0,4]} \subset G$

1) $\rho' < \rho$ $X_{\rho'} \subseteq X_\rho$

2) $0 \in X_0$

3) $x \in X_\rho \Rightarrow -x \in X_\rho$

4) ρ, ρ' s.t. $\rho + \rho' < 4$

$X_\rho + X_{\rho'} \subseteq X_{\rho+\rho'}$

5) $\rho \leq 1 \Rightarrow |X_{2\rho}| < 2^d \cdot |X_\rho|$

$|X_1| =$ size of S, $|S|$

$$B_\rho = (\chi_{X_\rho} / |X|_\rho) * \frac{\mathbb{1}_{X_\rho}}{|X|_\rho}$$

$\operatorname{supp} B_\rho \subset X_{2\rho}$

$\mu(S) = \frac{|S|}{|G|}$

$X_\rho = H$ subgroup G

Ex: (Bohr system)

suppose that $\Gamma = \{\gamma_1, \dots, \gamma_k\} \subset \hat{G}$, $k_1, \dots, k_k > 0$

$B_{k_1 \dots k_k}(\Gamma)$

$X_\rho = \{x \in G : |1 - \gamma_j(x)| < k_j \cdot \rho, \ j = 1 \dots k\}$

$k_j \cdot (\rho + \rho') \geq |1 - \gamma_j(x)| + |1 - \gamma_j(x)| \geq |\gamma_j(x) - \gamma_j(x')| = |1 - \gamma_j(x - x')|$

If $S = \operatorname{Bohr}_{k_1, \dots k_k}(\Gamma)$ $\dim S \leq 3k$ $\Uparrow$

$|S| \geq 8^{-k}_{k_1, \dots, k_k} |G|$

$S = (X_\rho)_{\rho \in [0,1]}$ $\dim S = d$, $\lambda \in (0,1)$

$\lambda S = (X_{\lambda\rho})_{\rho \in (0,4)}$ $|\lambda S| \geq (\frac{\lambda}{2})^d \cdot |S|$ $d : G \times G$ $X_\rho = \{x \in G : d(x, 0) < \rho\}$

Covering lemma:

suppose $\rho < \frac{1}{2}$, $X_{2\rho}$ may be covered by 2^{4d} translations of $X_{\rho/2}$

Metric entropy lemma:

suppose $\rho > 1$ G ~~would~~ may be covered by at most $\left(\frac{4}{\rho}\right)^d \cdot \mu(S)^{-1}$ translations of X_ρ.

$S = (X_\rho)$, $S' = (X_{\rho'})$

$S \wedge S' = (X_\rho \cap X'_\rho)$

lemma: $S \wedge S'$ is a B.S. $\dim(S \wedge S') \leq 4(d+d')$

$|S \wedge S'| \geq 2^{-3(d \cdot d')} \cdot \mu(S') \cdot |S|$

$X_{2\rho} \cap X'_{2\rho}$ by $2^{4(d+d')}$ sets of the form $(y + X_{\rho/2}) \cap (S' \cap X'_{\rho/2}) = T$

$t \mapsto t - t_0$ injection $T \to X_\rho \cap X_{\rho'}$

$t_0 \in T$, $t_0 \in y + X_{\rho/2}$ $t_0 = y + z$ $x \in X_{\rho/2}$

$t_0 \in y' + X'_{\rho/2}$ $t = y' + z'$ $z' \in X'_{\rho/2}$

$|T| \leq |X_\rho \cap X'_\rho|$

$|X_{2\rho} \cap X_{2\rho'}| \leq 2^{4(d+d')} \cdot |X_\rho \cap X_{\rho'}|$

regular B.S : $\dim(X_\rho) = d$

$1 - 10 \cdot d \cdot \varkappa \leq \frac{|X_1|}{|X_{1+\varkappa}|} \leq 1 + 10 \cdot d \cdot \varkappa$

Lemma: Suppose S is B.S. $\Rightarrow$ $\exists \lambda \in [\frac{1}{6}, \frac{5}{6}]$ s.t. λS is regular

$f : [0,1] \to \mathbb{R}$, $f(a) = \frac{1}{d} \log_2 |X_{2^a}|$

$f(1) - f(0) < 1$ $\exists a \in [\frac{1}{6}, \frac{5}{6}]$ s.t. $|f(a+x - f(a)| \leq 3|x|$

$|x| \leq \frac{1}{6}$ $(*)$

suppose not: $\underset{[\frac{1}{6}, \frac{5}{6}]}{\forall a}$ $\exists I_a$ s.t. $|I_a| \leq \frac{1}{6}$ s.t. $\int_{I_a} df > \int_{I_a} 3\,dx$

(f is monotone)

$[\frac{1}{6}, \frac{5}{6}] \subset \bigcup I_a$ $\Rightarrow$ $I_1 \cup I_2 \cup \ldots \cup I_n$ disjoint, $\sum |I_i| \geq \frac{1}{3}$

$1 \geq \int_0^1 df \geq \sum_i \int_{I_i} df > \sum \int_{I_i} 3\,dx \geq 1$

S - r. B.S. $\dim(S) = d$ $\varkappa = (0,1)$

$y \in X_\varkappa$ $\Rightarrow$ $\underset{x \in G}{\mathbb{E}} |\beta_1(x+y) - \beta_1(x)| \leq 20 d \varkappa$

$\mu_1 = \frac{\mathbb{1}_{X_1}}{|X_1|}$ $\beta_1 = \mu_1 * \mu_1$ $\varkappa > \frac{1}{10d}$

$|\mu_1(x+y) - \mu_1(x)| = 0$ unless $x \in X_{1+x} \setminus X_{1-x}$

$x+y \in X_1 \qquad x \notin X_1$

$x = (x+y) + (-y) \in X_1 + X_x \subseteq X_{1+x}$,

$x \in X_{1-x} \Rightarrow x+y \in X_{1-x} + X_x \subset X_1 \qquad x \notin X_1 \Rightarrow x \notin X_{1-x}$

$|X_{1+x} \setminus X_{1-x}| < 20 \cdot d \cdot x\, |X_1|$

$$\frac{|X_{1+x}|}{|X_1|} - \frac{|X_{1-x}|}{|X_1|} < 20 \cdot d \cdot x$$

$$\mathbb{E}_{x \in G} |\beta_1(x-y) - \beta_1(x)| = \mathbb{E}_{x \in G} |\mu_1 * \mu_1(x+y) - \mu_1 * \mu_1(x)|$$

$$\mathbb{E}_{x \in G} \left| \mathbb{E}_z \mu_1(z)\mu_1(x-y-z) - \mathbb{E}_z \mu(z)\mu_1(x-z) \right| = \mathbb{E}_z \mu_1(z) \mathbb{E}\, |\mu_1(x+y-z) - \mu_1(x-z)|$$

S. B. S.

$\varphi : L^\infty(G) \to L^\infty(G) \qquad \varphi_S f = f * \beta_1 \qquad ((\varphi_S f)^\wedge = \cancel{\hat{\varphi_S} \hat f}\ \hat\beta_1 \cdot \hat f)$

$\hat\beta_1 \geq 0$

$\|f\|_A = \|\varphi_S f\|_A + \|f - \varphi_S f\|_A$

<u>Lemma:</u> (almost invariance)

$f : G \to \mathbb{C} \qquad S$. r. B.S. $\quad \dim S = d \qquad x \in (0,1)$, $y \in X_x$

then $\quad |\varphi_S f(x+s) - \varphi_S f(x)| \leq 20 \cdot d \cdot x\, \|f\|_\infty$

$$\| \atop |\mathbb{E}_t f(t)(\beta_1(t-x+y) - \beta_1(t-x))|$$

sup

<u>23.03.2011</u>

<u>Proposition A:</u> $f : G \to \mathbb{R}$, $\|f\|_A \leq M \geq 1$

$d(f, \mathbb{Z}) < \frac{1}{4}$, S is regular B.S. s.t. $\dim S = d \geq 2$,

$\exists\ \varepsilon < \frac{1}{4} \Rightarrow \exists$ r. B.s. S' $\dim S' = d'$ s.t:

- $d' \leq 4d + \frac{64M^2}{\varepsilon^2}$
- $|S'| \geq \exp\left(-\left(\frac{c d M^4}{\varepsilon^4} \cdot \log \frac{dM}{\varepsilon}\right)\right) \cdot |S|$
- $\|\varphi_{S'} f\|_\infty \geq \|\varphi_S f\|_\infty - \varepsilon \,|\, d(\varphi_{S'} f, \mathbb{Z}) \leq d(f, \mathbb{Z}) + \varepsilon$

<u>Proposition B:</u>

$f : G \to \mathbb{R}$, $\|f\|_A \leq M \geq \frac{1}{2}$, $d(f, \mathbb{Z}) \leq e^{-CM^4}$. then there exists regular B.s. S s.t. 1) $\dim(S) \leq e^{CM^4}$, $\mu(S) \geq e^{-e^{CM^4}}$ $\|f\|_1$

$$\underbrace{\|\varphi_S f\|_\infty \geq e^{-CM^4}}_{\text{we'll call these conditions } (*)}$$

we'll call these conditions $(*)$.

we need to prove that:

$f: G \to \mathbb{R}$, $\|f\|_A \le M \geq 1$, $d(f, \mathbb{Z}) \le e^{-C_1 M^4}$, $\varepsilon = e^{-C_0 M^4}$

then $f = f_1 + f_2$

i) $\|f_1\|_A \le \|f\|_A - \frac{1}{2}$ or else $d(f_1, \mathbb{Z}) \le d(f, \mathbb{Z}) + \varepsilon$

(ii) $(f_1)_{\mathbb{Z}} = \sum_{j=1}^{L} \pm 1\!\!1_{x_j + H}$ $\quad L < e^{e^{C_0 M^4}}$

$\|f_2\|_A \le \|f\|_A - \frac{1}{2}$ or $d(f_2, \mathbb{Z}) \le 2 \cdot d(f, \mathbb{Z}) + \varepsilon$

$(*) \Rightarrow c = e^{-C_0 M^4}$ $\quad C_0 >> C$

$\exists\ S' = (x'_s)$ s.t. $\dim S' \le e^{c' M^4}$

$\mu(S') \geq e^{-e^{c' M^4}} \|f_{\mathbb{Z}}\|_1$

Moreover

$\|\varphi_{S'} f\|_\infty \geq \|\varphi_S f\|_\infty - \varepsilon > 2 e^{-CM^4}$

$d(\varphi_S f, \mathbb{Z}) \le d(f, \mathbb{Z}) + \varepsilon$

$f_1 = \varphi_{S'} f \qquad f_2 = f - \varphi_{S'} f$

$d(f_1, \mathbb{Z}) \le d(f, \mathbb{Z}) + \varepsilon$ by Δ ineq.

$d(f_2, \mathbb{Z}) \le 2 d(f, \mathbb{Z}) + \varepsilon$

we know $(f_1)_{\mathbb{Z}} \neq 0$ $\quad (d(f_1, \mathbb{Z}) \le d(f, \mathbb{Z}) + \varepsilon < 2 \cdot e^{-CM^4}$

$\|f_1\|_\infty \geq \|(f_1)_{\mathbb{Z}}\|_\infty - \varepsilon > 1/2 \Rightarrow \|f_1\|_A > \frac{1}{2}$

$\|f_2\|_A \le \|f\|_A - \frac{1}{2}$, $\|f\|_A = \|f_1\|_A + \|f_2\|_2$

suppose now that

$\|f_1\|_A > \|f\|_A - \frac{1}{2}$ (the else condition holds)

$\Rightarrow \|f_2\|_A < \frac{1}{2}$

$\|f_2\|_\infty < \frac{1}{2} \Rightarrow (f_2)_{\mathbb{Z}} = 0 \Rightarrow f_{\mathbb{Z}} = (\varphi_{S'} f)_{\mathbb{Z}}$

suppose

$x - x' \in X_{\frac{\varepsilon}{20dM}} \Rightarrow |f_{\mathbb{Z}}(x) - f_{\mathbb{Z}}(x')| = |(\varphi_{S'} f)_{\mathbb{Z}}(x) - (\varphi_{S'} f)_{\mathbb{Z}}(x')| \le$

$\le |\varphi_{S'} f(x) - \varphi_{S'} f(x')| + 2(d(f, \mathbb{Z}) + \varepsilon) \le 2d(f, \mathbb{Z}) + 3\varepsilon \le \frac{1}{10}$

$\left(\begin{array}{c} |\varphi_S f(x+y) - \varphi_S f(x)| < 20 d \cdot \varkappa \cdot \|f\|_\infty \\ y \in X_{\varkappa} \end{array} \right)$

$f_{\mathbb{Z}}(x) = f_{\mathbb{Z}}(x')$

$H = \langle X_{\frac{\varepsilon}{20dM}} \rangle$

$f_{\mathbb{Z}} = \sum^{L} \pm 1\!\!1_{x_j + H}$

We skip some part, but we'll show the estimation $L < e^{e^{c c_0 M^4}}$:

$$f_{\mathbb{Z}} = \sum_{i=1}^{L} \pm \mathbb{1}_{x_j + H}$$

$$L \cdot \|\mathbb{1}_H\|_1 \leq M \|f_{\mathbb{Z}}\|_1$$

$$\|\mathbb{1}_H\|_1 > \frac{\left|X_{\frac{\varepsilon}{20dr}}\right|}{|G|} \geq \left(\frac{\varepsilon}{40Md}\right)^d \cdot \overset{\mu(S')}{\frac{|X_1|}{|G|}}$$

(Recall: $\mu(S') \geq e^{-e^{c'M^4}} \|f_{\mathbb{Z}}\|_1$)