Improved Approximation Ratio for Strategyproof Facility Location on a Cycle

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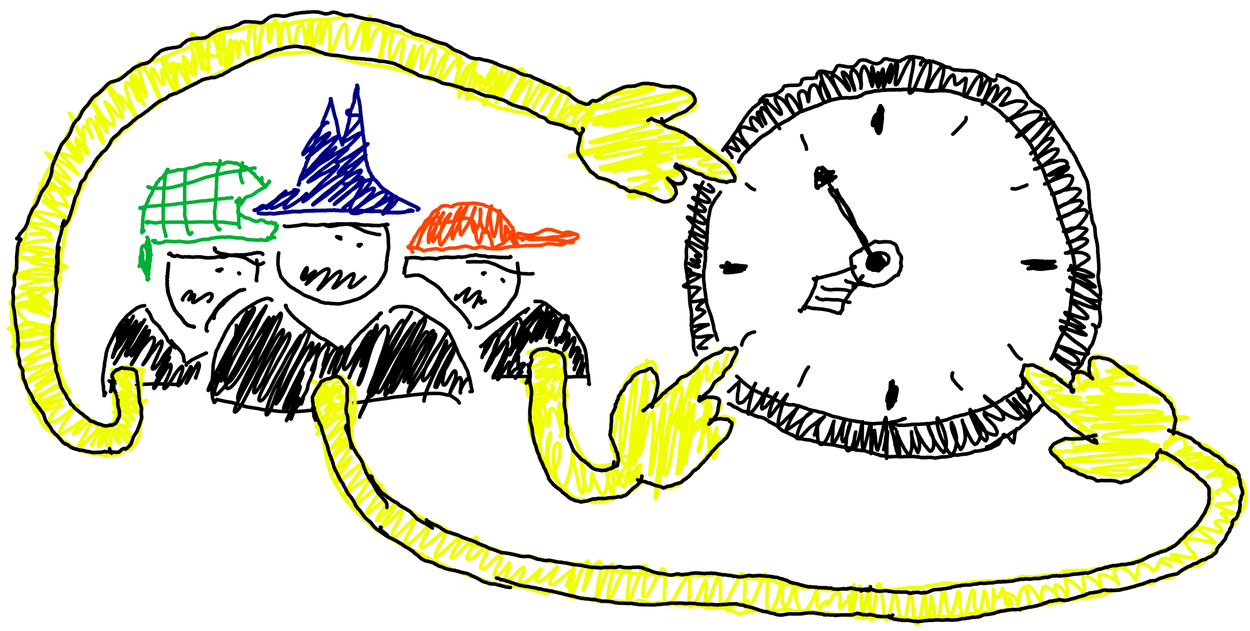
Facility Location (on a Cycle)

Facility location models problems where a decision is made based on the preferences of a set of agents, such as choosing a location for a library or *agreeing on a meeting time*. We study the problem from the mechanism design perspective, where a social planner, not knowing agents' preferences, must design a **mechanism** (a function mapping agents' votes to a decision). Agents observe the mechanism and vote strategically to minimize their own costs, while the social planner aims to minimize **social cost** (the sum of agents' costs).



A mechanism is **strategyproof** (SP) if no agent can decrease their cost by misreporting their vote. The quality of a mechanism is measured by its **approximation ratio**, defined as the worst-case ratio of the social cost of the mechanism outcome to the social cost of the optimal decision. Although the problem has been studied since the 1980s, design of optimal SP mechanisms is unknown when the underlying graph is neither a line segment nor a tree. Here, we focus on the case of a cycle graph G and propose a new best-performing mechanism for an odd number of agents ≥ 5.

Mixed Mechanism RD+PCD

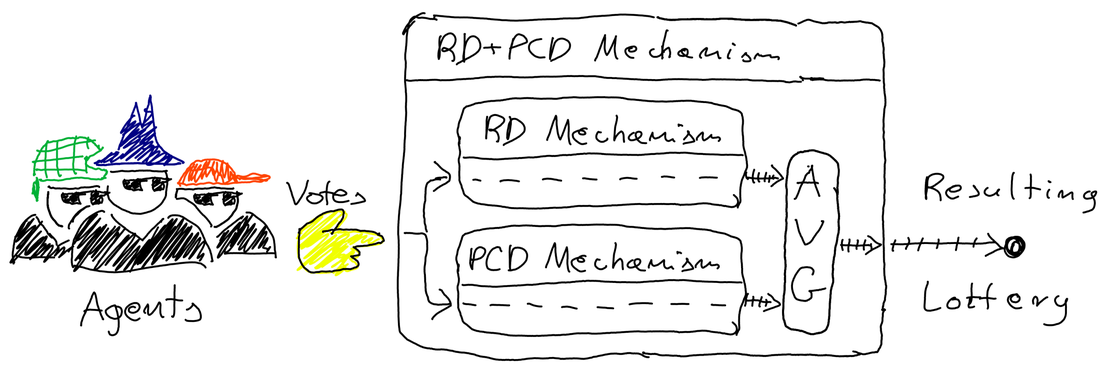
RD - Random Dictator; PCD - Proportional Circle Distance (Meir 2019)

Definition. The RD+PCD mechanism, for a given message profile s, returns a lottery that is the average of the lotteries returned by RD and PCD:

$$\forall v \in G: \quad \mathbb{P}((\text{RD+PCD})(s) = v) = \frac{\mathbb{P}(\text{RD}(s) = v) + \mathbb{P}(\text{PCD}(s) = v)}{2}$$

Immediate strengths of mixing:

- Preserves *strategyproofness*, neutrality, and anonymity
- Does not increase the approximation ratio compared to input mechanisms (upper bound 2), and can potentially improve it (possible up to 1.5)



Main Result

Theorem. For any set of n agents with an odd cardinality and a cycle graph G, approximation ratio of RD+PCD mechanism is upper-bounded by 1.75.

Comparison with previous work:

Lower Bounds		Upper Bounds		
By LP (Meir 2019)	Optimal SP Mechanism	**Our Experiments**	**Our Results**	Previous Work (RD mechanism)
$1.045 \leq$	?	$\mathbf{\leq 1.5}$	$\mathbf{\leq 1.75}$	$\leq 2 - 2/n$

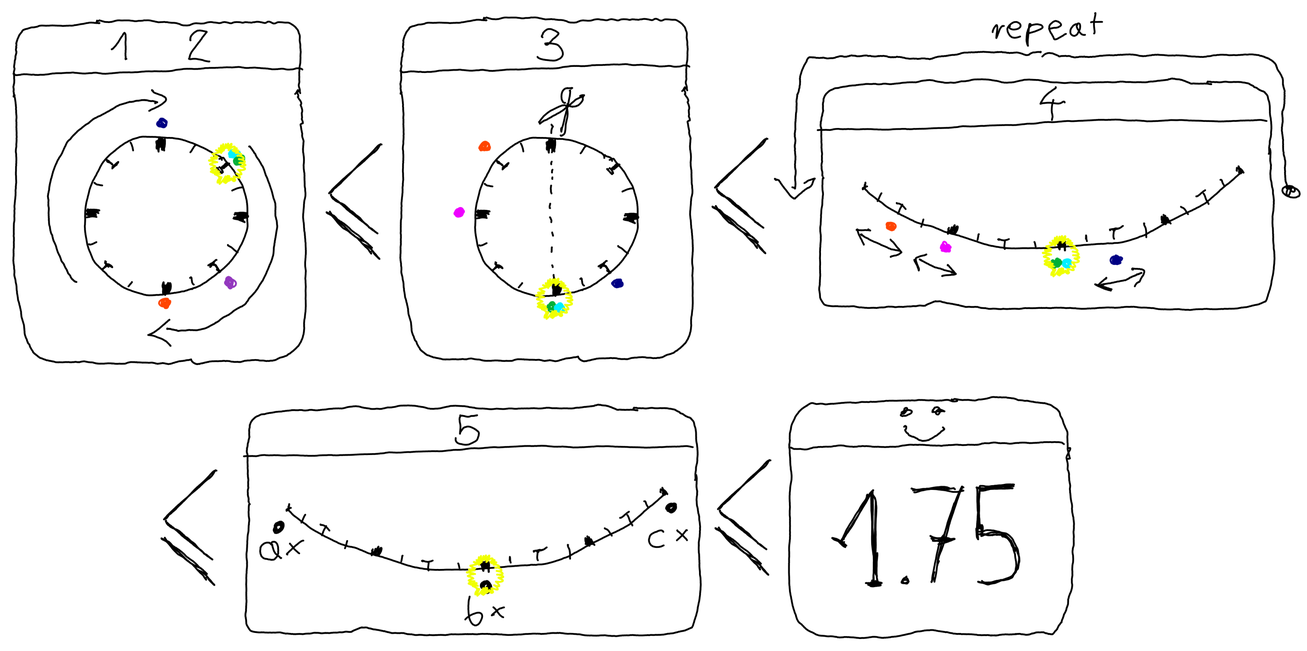
Bounding techniques

Convenience normalizations (without loss of generality):

1. Use neutrality to fix a reference vertex o of the cycle G for which the social cost is minimal.
2. Use anonymity to relabel agents with respect to o.

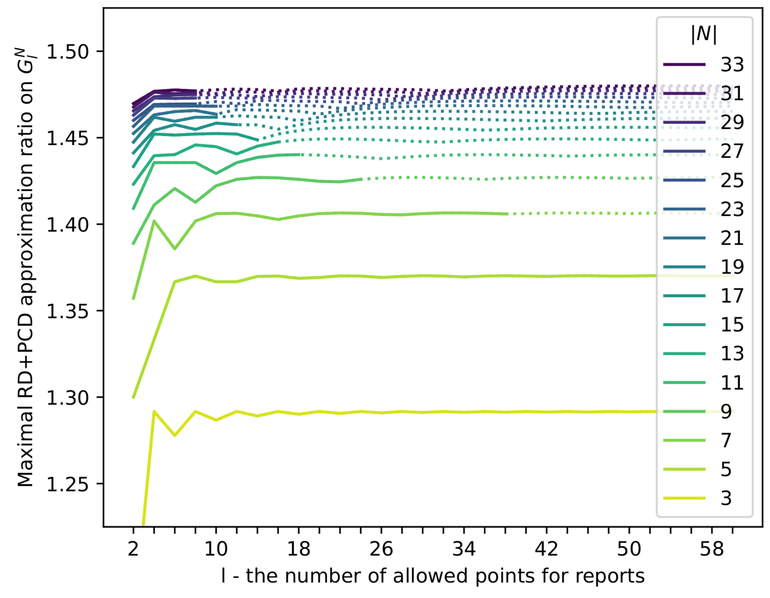
Main bounding steps:

3. Estimate cycle distances by line segment distances in the approximation ratio formula (conceptually, cut the cycle at the vertex antipodal to o).
4. Show monotonicity: repeatedly move votes of agent groups to extremal positions to obtain upper-bound.
5. Directly estimate the final expression to obtain the desired bound.



Experiments

Hypothesis. For any set of n agents with an odd cardinality and a cycle graph G, approximation ratio of RD+PCD mechanism is bounded from above by 1.5.



Methodology:

- Exhaustively test RD+PCD on a wide range of cycle graphs G_l with varying numbers of vertices l and set of agents N.
- Analyze how the approximation ratio changes with respect to both the number of agents and the graph size.
- Observe that the bound of 1.5 holds consistently.