Towards the Chen-Raspaud conjecture*

Katarzyna Łyczek† Maria Nazarczuk‡ Paweł Rzążewski§

Abstract

For an integer k, a homomorphism from a graph G to the Kneser graph $K(2k+1,k)$ is equivalent to assigning to each vertex of G a k-element subset of $\{1,\ldots,2k+1\}$ in a way that adjacent vertices receive disjoint subsets.

Chen and Raspaud [Discrete Mathematics, 2010] conjectured that for every $k \geq 2$, every graph G with maximum average degree less than $\frac{2k+1}{k}$ and no odd cycles with fewer than $2k+1$ vertices admits a homomorphism to $K(2k+1,k)$. They also showed that the statement is true for $k = 2$. In this note we confirm the conjecture for $k = 3$.

Keywords: graph homomorphism, fractional coloring, Chen-Raspaud conjecture

1 Introduction

Graph coloring is arguably one of the best studied graph problems and studying it has led to many exciting and deep results about the structure of graphs, including the celebrated four color theorem [1,2] or the notions of perfect [6,17] and χ-bounded graph classes [14].

Besides the classic graph coloring, many other variants are also studied [15]. For example for integers a, b, in $a{:}b$-coloring we assign to every vertex of a graph a b-element sets of colors among $\{1,\ldots,a\}$ and require that the sets assigned to adjacent vertices are disjoint. Such a coloring is motivated by some problems in scheduling [7], and is closely related to a well-studied graph parameter called the *fractional chromatic number* [13].

A convenient way of looking at the coloring problems is through the lens of graph homomorphisms. A homomorphism from a graph G to a graph H is a mapping $\varphi : V(G) \to V(H)$, such that if $uv \in E(G)$, then $\varphi(u)\varphi(v) \in E(H)$. We often refer to homomorphisms to H as H-colorings, and to the vertices of H as colors. The reason is that if H is K_k, i.e., the complete graph on k vertices, then homomorphisms to H are precisely proper k-colorings. Similarly, $a{:}b$-colorings are precisely homomorphisms to the *Kneser graph* $K(a,b)$. The vertex set of this graph consists of all b-element subsets of $\{1,\ldots,a\}$ and two sets are adjacent if and only if they are disjoint.

Some famous graphs are Kneser graphs. For example K_k is precisely $K(k,1)$ and the Petersen graph is $K(5,2)$. In general, Kneser graphs of the form $K(2k+1,k)$ are called *odd graphs* and they received some attention due to their interesting structure [4,11] and possible applications [3].

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†Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Poland, kl417711@students.mimuw.edu.pl

‡Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Poland, m.nazarczuk@student.uw.edu.pl

§Faculty of Mathematics and Information Science, Warsaw University of Technology, Warsaw, Poland and Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Poland, pawel.rzazewski@pw.edu.pl

Intuitively, if a graph is sparse, then it admits a proper coloring with few colors. This observation can be formalized as follows. For a graph G, its *maximum average degree* $\mathrm{mad}(G)$ is defined as the maximum average degree over all subgraphs of G. It is easy to show that if $\mathrm{mad}(G) < k$, then G can be properly colored with k colors (this was probably first observed by Szekeres and Wilf [16]).

Chen and Raspaud [5] studied the analogue of this result in the world of $(2k+1)$:k-coloring (i.e., homomorphisms to odd graphs $K(2k+1,k)$). One can observe that here imposing a bound on the maximum average degree is not sufficient. Indeed, it is known that if G admits a homomorphism to H, then $\text{odd-girth}(G) \geq \text{odd-girth}(H)$ [8], where $\text{odd-girth}(G)$ is the length of a shortest odd cycle in G. (For completeness, we define $\text{odd-girth}(G) := \infty$ if G is bipartite.) As it is known that $\text{odd-girth}(K(2k+1,k)) = 2k+1$ [12], every graph G with a homomorphism to $K(2k+1,k)$ must necessarily satisfy $\text{odd-girth}(G) \geq 2k+1$. Chen and Raspaud conjectured that adding this assumption is sufficient to obtain a homomorphism to $K(2k+1,k)$.

Chen-Raspaud Conjecture ([5])**.** Let $k \geq 2$ and let G be a graph with $\text{odd-girth}(G) \geq 2k+1$ and $\mathrm{mad}(G) < \frac{2k+1}{k}$. Then G admits a homomorphism to $K(2k+1,k)$.

It is known that, if true, this conjecture is best possible. Indeed, for every $k \geq 2$ there are graphs G with $\text{odd-girth}(G) \geq 2k+1$ and $\mathrm{mad}(G) = \frac{2k+1}{k}$ which do not have a homomorphism to $K(2k+1,k)$ [9].

Chen and Raspaud [5] proved the conjecture for $k = 2$ and all other cases remain wide open. In this paper we confirm the Chen-Raspaud conjecture for $k = 3$.

Theorem 1. *Let G be a graph with* $\text{odd-girth}(G) \geq 7$ *and* $\mathrm{mad}(G) < \frac{7}{3}$. *Then G admits a homomorphism to $K(7,3)$.*

Our proof uses the *discharging* method. For contradiction, we assume that Theorem 1 does not hold and consider a minimal counterexample G. First, in Section 3.1, we analyze the structure of G and show that certain substructures cannot appear there, as otherwise we would find a smaller counterexample. Then, in Section 3.2, we proceed to the discharging phase. Initially, each vertex of G receives a *charge* equal to its degree. Then we redistribute charges over vertices; in this phase the total sum of charges remains the same. Then we analyze the final charges. We show that since certain structures do not appear in G, every vertex has final charge at least $\frac{7}{3}$. However, this is a contradiction with the assumption on $\mathrm{mad}(G)$. Consequently, a counterexample to Theorem 1 cannot exist.

Some claims, marked with (⚙), required tedious case analysis. We wrote a simple Python program that verifies these claims. The code is available on the third author's website [10].

2 Notation and preliminaries

All graphs considered in this paper are simple and finite. For a graph G, we use $V(G), E(G)$ to denote its vertex and edge set, respectively.

A walk in a graph G is a sequence $v_0, v_1, \ldots, v_k$ of (not-necessarily distinct) vertices in which consecutive vertices are adjacent. The *length* of the walk is k, i.e., the number of vertices minus 1. A u-v-walk is a walk whose first vertex is u and the last vertex is v.

A walk of length at least 2, where (i) all vertices are mutually distinct, with possible exception that the first and the last vertex are the same, and (ii) all internal vertices are of degree 2 are the endvertices are of degree other than 2 is called a *thread*. In particular, an edge joining two vertices of degree other than 2 is a thread of length 1 (with no internal vertices). A thread in which the first and the last vertex are the same is called *pinched*. If u and v and the endvertices of a thread, we say that they are *relatives* (or *related*). For a vertex v of degree at least 3, by $\text{star}(v)$ we denote the set consisting of v and all internal vertices of the threads containing v.

Let v be a vertex of degree $k \geq 3$ that does not belong to any pinched thread. Clearly it belongs to k threads. If the lengths of these threads are $a_1 \leq a_2 \leq \ldots \leq a_k$, respectively, then we say that v is an $(a_1, a_2, ..., a_k)$*-vertex*. Sometimes we also allow that a_i is an integer decorated with superscript '+', which indicates that the particular thread is of length at least a_i.

The *average degree* $\text{ad}(G)$ of a given graph G is defined as $\text{ad}(G) = \frac{2|E(G)|}{|V(G)|}$. Recall that the *maximum average degree* $\text{mad}(G)$ of G is $\max\left\{\frac{2|E(H)|}{|V(H)|} \mid H \subseteq G\right\}$.

In this paper we are interested in the Kneser graph $K(7,3)$. We use the convention that $V(K(7,3)) = \binom{[7]}{3}$, i.e., it consists of 3-element subsets of $\{0, 1, \ldots, 6\}$. These subsets are called *colors* and we call a homomorphism to $K(7,3)$ a $K(7,3)$*-coloring*.

Consider a homomorphism $f : G \to K(7,3)$ and let $v_0, v_1, \ldots, v_i$ be a walk in G. We observe that $f(v_0), f(v_1), \ldots, f(v_i)$ is a walk in $K(7,3)$. Thus, if for some $y \in \binom{[7]}{3}$ the graph $K(7,3)$ does not have an $f(v_0)$-y walk of length i, certainly we have $f(v_i) \neq y$. This justifies the following definitions (where A stands for *allowed* and F stands for *forbidden*).

Definition 2. For $x \in \binom{[7]}{3}$ and $i \in \mathbb{N}_+$ we define sets:

$$A_i(x) = \{y \in \binom{[7]}{3} \mid \text{there exists an } x\text{-}y\text{-walk of length } i \text{ in } K(7,3)\},$$

$$F_i(x) = \binom{[7]}{3} \setminus A_i(v).$$

The following observation is immediate, see Table 1 and recall that $K(7,3)$ is vertex-transitive.

Observation 3. *For every $x \in \binom{[7]}{3}$ we have:*

$$\begin{aligned}
|A_1(x)| &= 4 & |F_1(x)| &= 31\\
|A_2(x)| &= 13 & |F_2(x)| &= 22\\
|A_3(x)| &= 22 & |F_3(x)| &= 13\\
|A_4(x)| &= 31 & |F_4(x)| &= 4\\
|A_5(x)| &= 34 & |F_5(x)| &= 1\\
|A_i(x)| &= 35 & |F_i(x)| &= 0 \quad \textit{for every } i \geq 6.
\end{aligned}$$

Let us conclude this section with a useful lemma, see Figure 1 for the illustration.

Lemma 4. *Let G be a graph and let v be an (a_1, a_2, a_3)-vertex that does not belong to any pinched thread, denote the relatives of v by v_1, v_2, v_3. Let G' be obtained from G by removing $\text{star}(v)$, and adding a new thread of length $a_1 + a_2$ with endvertices v_1 and v_2, and a new thread of length $a_1 + a_3$ with endvertices v_1 and v_3. If $\text{mad}(G) < \frac{7}{3}$, then $\text{mad}(G') < \frac{7}{3}$.*

Proof. For contradiction, suppose that $\text{mad}(G') \geq \frac{7}{3}$. Let F' be a subgraph of G' with the largest average degree, and denote $m := |E(F')|$ and $n := |V(F')|$. Thus we have

$$\frac{2m}{n} = \text{ad}(F') = \text{mad}(G') \geq \frac{7}{3}.$$

Let us start with a technical claim.

Claim 4.1. *Let $q \geq 1$ and F be a graph with $n - q$ vertices and $m - q$ edges. Then $\text{ad}(F) > \text{ad}(F')$.*

i	$A_i(\{0,1,2\})$
1	{3, 4, 5}, {3, 4, 6}, {3, 5, 6}, {4, 5, 6}
2	{0, 1, 4}, {0, 2, 4}, {1, 2, 6}, {0, 1, 3}, {1, 2, 3}, {0, 2, 3}, {0, 1, 6}, {0, 2, 6}, {0, 1, 5}, {1, 2, 5}, {0, 2, 5}, {0, 1, 2}, {1, 2, 4}
3	{3, 5, 6}, {0, 3, 6}, {4, 5, 6}, {1, 3, 6}, {1, 4, 5}, {2, 4, 5}, {2, 3, 6}, {0, 5, 6}, {1, 5, 6}, {0, 3, 5}, {2, 5, 6}, {3, 4, 6}, {1, 3, 5}, {2, 3, 5}, {0, 4, 6}, {0, 3, 4}, {1, 4, 6}, {1, 3, 4}, {3, 4, 5}, {2, 3, 4}, {2, 4, 6}, {0, 4, 5}
4	{0, 1, 3}, {0, 1, 6}, {0, 3, 6}, {1, 2, 5}, {1, 3, 6}, {1, 4, 5}, {2, 4, 5}, {0, 2, 4}, {2, 3, 6}, {0, 5, 6}, {1, 5, 6}, {0, 1, 5}, {0, 1, 2}, {0, 3, 5}, {1, 2, 4}, {2, 5, 6}, {1, 3, 5}, {0, 2, 3}, {2, 3, 5}, {0, 2, 6}, {0, 4, 6}, {0, 1, 4}, {0, 3, 4}, {1, 2, 6}, {1, 2, 3}, {1, 4, 6}, {1, 3, 4}, {2, 3, 4}, {0, 2, 5}, {2, 4, 6}, {0, 4, 5}
5	{3, 5, 6}, {0, 1, 3}, {0, 1, 6}, {0, 3, 6}, {4, 5, 6}, {1, 2, 5}, {1, 3, 6}, {1, 4, 5}, {2, 4, 5}, {0, 2, 4}, {2, 3, 6}, {0, 5, 6}, {1, 5, 6}, {0, 1, 5}, {0, 3, 5}, {1, 2, 4}, {2, 5, 6}, {3, 4, 6}, {1, 3, 5}, {0, 2, 3}, {2, 3, 5}, {0, 2, 6}, {0, 4, 6}, {0, 1, 4}, {0, 3, 4}, {1, 2, 6}, {1, 2, 3}, {1, 4, 6}, {1, 3, 4}, {3, 4, 5}, {2, 3, 4}, {0, 2, 5}, {2, 4, 6}, {0, 4, 5}
$i \geq 6$	{3, 5, 6}, {0, 1, 3}, {0, 1, 6}, {0, 3, 6}, {4, 5, 6}, {1, 2, 5}, {1, 3, 6}, {1, 4, 5}, {2, 4, 5}, {0, 2, 4}, {2, 3, 6}, {0, 5, 6}, {1, 5, 6}, {0, 1, 5}, {0, 1, 2}, {0, 3, 5}, {1, 2, 4}, {2, 5, 6}, {3, 4, 6}, {1, 3, 5}, {0, 2, 3}, {2, 3, 5}, {0, 2, 6}, {0, 4, 6}, {0, 1, 4}, {0, 3, 4}, {1, 2, 6}, {1, 2, 3}, {1, 4, 6}, {1, 3, 4}, {3, 4, 5}, {2, 3, 4}, {0, 2, 5}, {2, 4, 6}, {0, 4, 5}

Table 1: The sets $A_i(\{0,1,2\})$.

Proof of Claim. For contradiction, suppose the opposite.

$$\begin{aligned} \mathrm{ad}(F') &\geq \mathrm{ad}(F) \\ \frac{2m}{n} &\geq \frac{2(m-q)}{n-q} \\ n &\geq m. \end{aligned}$$

Consequetly, we obtain

$$2 \geq \frac{2m}{n} = \mathrm{ad}(F') \geq \frac{7}{3},$$

a contradiction. ⌟

Claim 4.1 immediately implies the following.

Claim 4.2. *F' has no vertices of degree* 1.

Proof of Claim. Suppose the opposite, and let u be a vertex of degree 1 in F'. Then the graph $F := F' - u$ has $n-1$ vertices and $m-1$ edges, so $\mathrm{ad}(F) > \mathrm{ad}(F')$ by Claim 4.1. This contradicts the choice of F'. ⌟

Claim 4.3. *F' contains at least one internal vertex of each of the two new threads v_1.*

Proof of Claim. Otherwise F' is isomorphic to a subgraph of G and therefore $\mathrm{mad}(G) \geq \mathrm{ad}(F') \geq \frac{7}{3}$, a contradiction. ⌟

Given Claim 4.2 and Claim 4.3 we obtain that F' contains either exactly one of the two new threads or both of them. In the former case, G contains a subgraph isomorphic to F', a contradiction. Therefore, F' must contain all newly added vertices, along with v_1, v_2, v_3.

Let F be the subgraph of G obtained from F' by removing all newly added vertices, and restoring v along with its three threads (i.e., star(v) with all incident edges). We observe that

$$\begin{aligned} |E(F)| &= m - a_1 \\ |V(F)| &= n - a_1. \end{aligned}$$

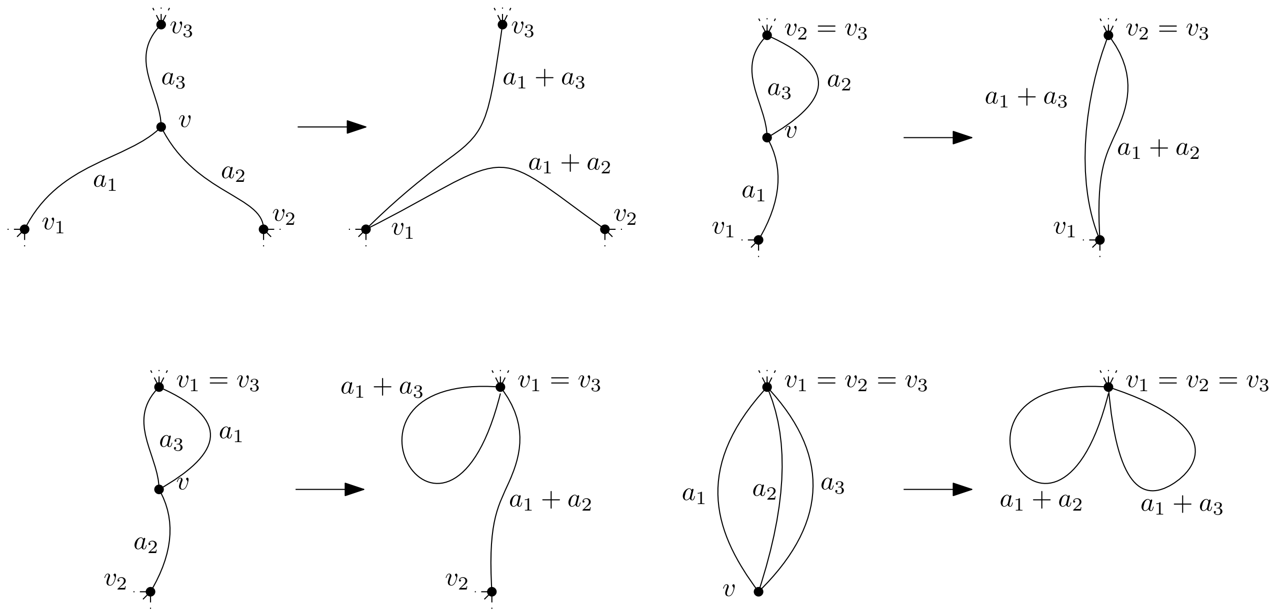

Figure 1: The operation in Lemma 4. Vertices v_1, v_2, v_3 do not need to be distinct and they may have neighbors not depicted in the figure.

By Claim 4.1 we conclude that $\text{mad}(G) \geq \text{ad}(F) > \text{ad}(F') = \text{mad}(G')$, a contradiction. Consequently, we conclude that $\text{mad}(G') < \frac{7}{3}$. □

3 Proof of Theorem 1

For contradiction suppose that Theorem 1 does not hold, i.e., there are graphs with maximum average degree smaller that $\frac{7}{3}$ and odd girth at least 7, which do not admit a homomorphism to $K(7,3)$. Among all such counterexamples let G be one with the fewest vertices of degree at least 3. If there is more than one such graph, we choose an arbitrary one with the minimum number of edges.

Note that every subgraph G' of G satisfies $\text{mad}(G') \leq \text{mad}(G) < \frac{7}{3}$ and $\text{odd-girth}(G') \geq \text{odd-girth}(G) \geq 7$, and thus, by the choice of G, admits a homomorphism to $K(7,3)$.

3.1 Forbidden configurations

In this section we present a series of technical claims in which we analyze the structure of the graph G.

Claim 1.4. *G has no vertices of degree less than* 2.

Proof of Claim. For contradiction, suppose that G contains a vertex v of degree at most 1. As K_1 is trivially $K(7,3)$-colorable, we may assume that $G \setminus \{v\}$ has at least one vertex. Let u be the unique neighbor of v (if $\deg v = 1$) or an arbitrary vertex of $G \setminus \{v\}$.

By the minimality of G, there is a homomorphism φ from $G \setminus \{v\}$ to $K(7,3)$. We can easily extend it to a homomorphism from G to $K(7,3)$ by mapping v to an arbitrary vertex from $A_1(\varphi(u))$, see Observation 3. This contradicts the choice of G. ⌟

Claim 1.5. *G has no threads of length at least 6.*

Proof of Claim. For contradiction, suppose that G has a thread of length $\ell \geq 6$. Let u and v be the endvertices of such a thread, and let B denote the set of its internal vertices.

By the minimality of G, there is a homomorphism φ from $G \setminus B$ to $K(7,3)$. By Observation 3 we have $\varphi(v) \in A_\ell(\varphi(u))$ (as $(F_\ell(\varphi(u)) = \emptyset)$. Consequently, there is a $\varphi(u)$-$\varphi(v)$-walk in $K(7,3)$ of length ℓ. We can extend φ by mapping consecutive vertices of the thread to the consecutive vertices of this walk, obtaining a homomorphism from G to $K(7,3)$, a contradiction. ⌟

Claim 1.6. *G has no pinched threads.*

Proof of Claim. For contradiction, suppose that G has a pinched thread of length ℓ and let v be its unique vertex of degree at least 3. Clearly $\ell \geq 3$. Since odd-girth$(G) > 5$, we observe that $\ell \notin \{3,5\}$. By Claim 1.5 we conclude that $\ell < 6$.

Thus $\ell = 4$, denote its consecutive vertices by v, u, w, z, v. By the minimality of G there exists a homomorphism φ from $G \setminus \{u, w, z\}$ to $K(7,3)$. Let y be an arbitrary element of $A_1(\varphi(v))$, it exists as $A_1(\varphi(v)) \neq \emptyset$. We can extend φ by mapping u and z to y, and mapping w to $\varphi(v)$. This way we obtain a homomorphism from G to $K(7,3)$, a contradiction. ⌟

In the remainder of this section we analyze possible types of vertices of degree at least 3 that might appear in G. In particular, we show that certain vertices cannot appear. We group these forbidden vertices into three groups.

Vertices excluded by the colorings of their relatives. First, we show that in some cases, for any mapping of the relatives of a vertex v, we can extend this mapping to v and all threads containing v.

Claim 1.7. *Let $k \geq 3$ and let $a_1 \leq \ldots \leq a_k$ be positive integers, such that for $x \in \binom{[7]}{3}$ we have $\sum_{i=1}^{k} |F_{a_i}(x)| < 35$. Then G does not contain any $(a_1, a_2, ..., a_k)$-vertex.*

Proof of Claim. For contradiction suppose that G contains an $(a_1, a_2, ..., a_k)$-vertex v. Let $v_1, v_2, \ldots, v_k$ be the relatives of v, such that the v-v_i-thread has length a_i (here we do not assume that v_i's are pairwise distinct). By the minimality of G, there exists a homomorphism φ from $G \setminus \text{star}(v)$ to $K(7,3)$.

Since for all $x, y \in \binom{[7]}{3}$ and all i it holds that $|F_i(x)| = |F_i(y)|$, we have $\sum_{i=1}^{k} |F_{a_i}(\varphi(v_i))| < 35 = |\binom{[7]}{3}|$. Consequently, $\bigcap_{i=1}^{k} A_{a_i}(\varphi(v_i)) = \binom{[7]}{3} \setminus \bigcup_{i=1}^{k} F_{a_i}(\varphi(v_i)) \neq \emptyset$. Pick some $x \in \bigcap_{i=1}^{k} A_{a_i}(\varphi(v_i))$. We can extend φ to a homomorphism from G to $K(7,3)$ by mapping v to x and the internal vertices of each v-v_i-thread to the appropriate vertices of the x-$\varphi(v_i)$ walk of length a_i in $K(7,3)$. This contradicts the choice of G. ⌟

Note that if for all i it holds that $a_i \leq a'_i$, then $\sum_{i=1}^{k} |F_{a'_i}(x)| \leq \sum_{i=1}^{k} |F_{a_i}(x)|$. Thus comparing Claim 1.7 with Observation 3 we immediately obtain the following.

Claim 1.8. *The vertices of the following types do not appear in G:*

- $(1,5,5)$,
- $(2,4^+,4^+)$,
- $(3^+,3^+,4^+)$,
- $(1,5,5,5)$,
- $(2,4^+,4^+,4^+)$,
- $(3,3,4^+,4^+)$,
- $(2,4^+,4^+,5,5)$,
- $(3^+,3^+,4^+,5,5)$,
- $(4^+,5,5,5,5,5)$.

Vertices excluded by the colorings of subsets of their relatives. Note that the assumptions of Claim 1.7 were too strong for our application. First, we do not use the fact that some sets $F_{a_i}(x_i)$ may overlap and thus the sum of their union can be smaller than the sum of sizes. Second, we do not really need to know that *any* mapping of relatives of v can be extended to include $\text{star}(v)$, but we only care about mappings that can appear in the homomorphism from the rest of the graph to $K(7,3)$. Indeed, for some $a_1, \ldots, a_k$ we are able to show that either some mapping of the relatives of an $(a_1, \ldots, a_k)$-vertex v does not have to be considered, or we have a smaller counterexample, which contradicts the choice of G.

The intuition behind the following claim is that longer threads enforce fewer restrictions on the colors of their endvertices. Thus possibly after removing the longest thread containing v and finding a homomorphism from the rest of the graph to $K(7,3)$, we can extend this mapping to the mapping of G.

Claim 1.9. *Let $k \geq 3$ and let $a_1 \leq \ldots \leq a_k$ be positive integers, such that the following holds: for all $x_1, \ldots, x_k \in \binom{[7]}{3}$ we have $\bigcap_{i=1}^{k-1} A_{a_i}(x_i) \neq \emptyset$ if and only if $\bigcap_{i=1}^{k} A_{a_i}(x_i) \neq \emptyset$. Then G does not contain any $(a_1, \ldots, a_k)$-vertex.*

Proof of Claim. For contradiction, suppose that G contains an $(a_1, \ldots, a_k)$-vertex v. Let $v_1, \ldots, v_k$ be the relatives of v, such that the v-v_i-thread is of length a_i.

Let G' be obtained from G by deleting all internal vertices and edges of the v-v_k-thread, i.e., the one of length a_k. By the minimality of G, there is a homomorphism φ' from G' to $K(7,3)$. In particular, $\varphi'(v) \in \bigcap_{i=1}^{k-1} A_{a_i}(\varphi'(v_i))$. Thus, by our assumption, there is $x \in \bigcap_{i=1}^{k} A_{a_i}(\varphi'(v_i))$ (note that we do not claim that $x = \varphi'(v)$).

Let us define $\varphi : V(G) \to \binom{[7]}{3}$ as follows. For every vertex $u \in V(G) \setminus \text{star}(v)$ we define $\varphi(u) = \varphi'(u)$. Furthermore, we set $\varphi(v) = x$. Finally, we map the internal vertices of each v-v_i-thread to the appropriate vertices of the x-$\varphi(v_i)$ walk of length a_i in $K(7,3)$. It is straightforward to verify that φ is a homomorphism from G to $K(7,3)$. This contradicts the choice of G. ⌟

Using computer search we verify that some sequences satisfy the assumptions of Claim 1.9.

Claim 1.10 (⚙). *The vertices of the following types do not appear in G:*

- $(1,2,5)$,
- $(1,3,5)$,
- $(1,4,5)$,
- $(2,2,4)$,
- $(2,2,5)$,
- $(2,3,4)$,
- $(2,3,5)$,
- $(3,3,3)$,
- $(1,4,5,5)$,
- $(2,3,5,5)$.

Vertices of types $(1,3,4)$, $(1,4,4)$, and $(2,3,3)$. Finally let us focus on vertices of types $(1,3,4)$, $(1,4,4)$, and $(2,3,3)$; note that they do not satisfy Claim 1.9. The way we deal with them is very similar to Claim 1.9: again we show that if such a vertex exists in G, then G is not a minimal counterexample. However, this time the smaller counterexample we obtain is *not* a subgraph of G. In particular, this is the only place where we use the fact that G was assumed to minimize the number of vertices of degree at least 3.

Claim 1.11. *Let $a_1 \leq a_2 \leq a_3$ be positive integers. Suppose that for every $x_1, x_2, x_3 \in \binom{[7]}{3}$ such that $x_1 \in A_{a_1+a_2}(x_2) \cap A_{a_1+a_3}(x_3)$ it holds that $\bigcap_{i=1}^{3} A_{a_i}(x_i) \neq \emptyset$. Then G does not contain any (a_1, a_2, a_3)-vertex.*

Proof of Claim. For contradiction, suppose that G contains an (a_1, a_2, a_3)-vertex v. Let v_1, v_2, v_3 be the relatives of v, such that the v-v_i-thread is of length a_i. Note that v_1, v_2, v_3 are not necessarily distinct.

Let G' be obtained from G as in Lemma 4, i.e., by removing $\mathrm{star}(v)$, and adding a new thread of length $a_1 + a_2$ with endvertices v_1 and v_2, and a new thread of length $a_1 + a_3$ with endvertices v_1 and v_3. By Lemma 4 we know that $\mathrm{mad}(G') < \frac{7}{3}$. Furthermore, clearly $\text{odd-girth}(G') \geq \text{odd-girth}(G) > 5$. Finally, G' has fewer vertices of degree at least 3 than G. Thus, by the minimality of G, there is a homomorphism φ' from G' to $K(7,3)$. Note that this means that $\varphi(v_1) \in A_{a_1+a_2}(\varphi(v_2)) \cap A_{a_1+a_3}(\varphi(v_3))$. By the assumption of the claim there exists $x \in \bigcap_{i=1}^{3} A_{a_i}(\varphi(v_i))$. Thus for every $i \in \{1,2,3\}$ there is an x-$\varphi(v_i)$-walk in $K(7,3)$ of length a_i. Now we can define a homomorphism φ from G to $K(7,3)$ in a way analogous to the proof of Claim 1.9. This contradicts the choice of G. ⌟

Again, using computer search we verify that sequences $(1,3,4)$, $(1,4,4)$, and $(2,3,3)$ satisfy the assumptions of Claim 1.11.

Claim 1.12 (⚙). *The vertices of the following types do not appear in G:* $(1,3,4)$, $(1,4,4)$, $(2,3,3)$.

Combining Claim 1.5, Claim 1.8, Claim 1.10, and Claim 1.12, we obtain that the only types of vertices of degree at least 3 that might possibly appear in G are as listed in Table 2.

3.2 Discharging

Initially, each vertex v receives a charge $\mathfrak{w}(v)$ equal to its degree $\deg v$. We apply the following discharging rule: each vertex v of degree 2 receives a charge $\frac{1}{6}$ from the endvertices of the unique thread containing v; recall that these vertices are distinct by Claim 1.6. Note that during this process the total charge assigned to the graph remains the same. We will show that after the discharging each vertex has charge $\mathfrak{w}^*(v) \geq \frac{7}{3}$, which will lead to a contradiction as follows:

$$\frac{7}{3} \leq \frac{\sum_{v \in V(G)} \mathfrak{w}^*(v)}{|V(G)|} = \frac{\sum_{v \in V(G)} \mathfrak{w}(v)}{|V(G)|} = \frac{\sum_{v \in V(G)} \deg v}{|V(G)|} = \frac{2|E(G)|}{|V(G)|} \leq \mathrm{mad}(G) < \frac{7}{3}. \tag{1}$$

So the only thing left is to analyze the final charges $\mathfrak{w}^*(\cdot)$. Let v be an arbitrary vertex of G. The analysis is split into cases, depending on the degree of v. Recall that by Claim 1.4 we always have $\deg v \geq 2$.

Case $\deg v = 2$**.** In this case v loses no charge and receives total charge $2 \cdot \frac{1}{6}$, thus we have

$$\mathfrak{w}^*(v) = \mathfrak{w}(v) + 2 \cdot \frac{1}{6} = \deg v + \frac{1}{3} = 2 + \frac{1}{3} = \frac{7}{3}.$$

All other types of vertices only lose charge. From now on, assume that $\deg v \geq 3$. For $i \in \{1,\ldots,5\}$, let d_i be the number of threads of length i containing v. By Claim 1.5 and

degree	possible types of vertices
3	$(1,1,1^+), (1,2,2), (1,2,3), (1,2,4), (1,3,3), (2,2,2), (2,2,3)$
4	all except $(1,4^+,5,5), (2,3,5,5), (2,4^+,4^+,4^+), (3^+,3^+,4^+,4^+)$
5	all except $(2,4^+,4^+,5,5)$, $(3^+,3^+,4^+,5,5)$
6	all except $(4^+,5,5,5,5,5)$
≥ 7	all

Table 2: Possible types of vertices v in G. We only list vertices whose threads are all of length at most 5, as by Claim 1.5 there are no threads of length at least 6.

Claim 1.6 we have $\deg v = \sum_{i=1}^{5} d_i$ and so $d_5 = \deg v - \sum_{i=1}^{4} d_i$. The final charge at v is

$$\begin{aligned}\mathfrak{w}^*(v) =& \mathfrak{w}(v) - \frac{1}{6} \cdot \sum_{i=1}^{5} d_i(i-1) = \deg v - \frac{1}{6}(d_2 + 2d_3 + 3d_4 + 4d_5) \\ =& \deg v - \frac{1}{6}\left(d_2 + 2d_3 + 3d_4 + 4(\deg v - d_1 - d_2 - d_3 - d_4)\right) \\ =& \frac{1}{3}\deg v + \frac{1}{6}(4d_1 + 3d_2 + 2d_3 + d_4)\end{aligned} \tag{2}$$

Case $\deg v = 3$. By Claims 1.8, 1.10, and 1.12 the only possible vertices of degree 3 are of types: $(1,1,a), (1,2,b), (1,3,3), (2,2,c)$ for $a \in \{1,2,3,4,5\}$, $b \in \{2,3,4\}$, $c \in \{2,3\}$; see also Table 2. A direct check shows that this implies that $4d_1 + 3d_2 + 2d_3 + d_4 \geq 8$. By (2) this gives us

$$\mathfrak{w}^*(v) = \frac{1}{3}\deg v + \frac{1}{6}(4d_1 + 3d_2 + 2d_3 + d_4) \geq \frac{3}{3} + \frac{4}{3} = \frac{7}{3}.$$

Case $\deg v = 4$. By Claim 1.8 and Claim 1.10 we know that v is neither a $(1,4^+,5,5)$-, a $(2,3,5,5)$-, a $(2,4^+,4^+,4^+)$-, nor a $(3^+,3^+,4^+,4^+)$-vertex. We claim that this implies that

$$4d_1 + 3d_2 + 2d_3 + d_4 \geq 6. \tag{3}$$

Consider the cases. First suppose that $d_1 \geq 1$. Since v is not a $(1,4^+,5,5)$-vertex, one of the following cases must hold:

$$d_1 \geq 2, \quad \text{or} \quad d_1 = 1 \text{ and } (d_2 + d_3) \geq 1, \quad \text{or } d_1 = 1 \text{ and } d_4 \geq 2.$$

We observe that in each of them (3) holds. So from now on we assume that $d_1 = 0$. Now suppose that $d_2 \geq 1$. Since v is neither a $(2,3,5,5)$-vertex nor a $(2,4^+,4^+,4^+)$-vertex, we observe that one of the following cases must hold:

$$d_2 \geq 2, \quad \text{or} \quad d_2 = 1 \text{ and } d_3 \geq 2, \quad \text{or } d_2 = 1 \text{ and } d_3 = 1 \text{ and } d_4 \geq 1.$$

Again, in each of them (3) holds. Now we note that if $d_1 = d_2 = 0$, then, since v is not a $(3^+,3^+,4^+,4^+)$-vertex, we must have $d_3 \geq 3$, which means that (3) holds.

Combining (2) with (3) gives us

$$\mathfrak{w}^*(v) = \frac{1}{3}\deg v + \frac{1}{6}(4d_1 + 3d_2 + 2d_3 + d_4) \geq \frac{4}{3} + \frac{3}{3} = \frac{7}{3}.$$

Case $\deg v = 5$. By Claim 1.8 we know that v is neither a $(2,4^+,4^+,5,5)$- nor a $(3^+,3^+,4^+,5,5)$-vertex. A direct check shows that this implies that $4d_1 + 3d_2 + 2d_3 + d_4 \geq 4$. By (2) this gives us

$$\mathfrak{w}^*(v) = \frac{1}{3}\deg v + \frac{1}{6}(4d_1 + 3d_2 + 2d_3 + d_4) \geq \frac{5}{3} + \frac{2}{3} = \frac{7}{3}.$$

Case $\deg v = 6$. By Claim 1.8 we know that v is not a $(4^+,5,5,5,5,5)$-vertex, i.e., either $d_5 \leq 4$ or $d_5 = 5$ and $d_4 = 0$. Consequently, we have $4d_1 + 3d_2 + 2d_3 + d_4 \geq 2$. By (2) this gives us

$$\mathfrak{w}^*(v) = \frac{1}{3}\deg v + \frac{1}{6}(4d_1 + 3d_2 + 2d_3 + d_4) \geq \frac{6}{3} + \frac{1}{3} = \frac{7}{3}.$$

Case $\deg v \geq 7$. By (2) we have

$$\mathfrak{w}^*(v) = \frac{1}{3}\deg v + \frac{1}{6}(4d_1 + 3d_2 + 2d_3 + d_4) \geq \frac{1}{3}\deg v \geq \frac{7}{3}.$$

Summing up, by (1) we obtain a contradiction. This means that a hypothetical counterexample to Theorem 1 cannot exist. This completes the proof.

4 Conclusion

Our Theorem 1, combined with the theorem of Chen and Raspaud [5], implies that the Chen-Raspaud conjecture holds for $k \in \{2, 3\}$. An obvious direction of further research is to consider other values of k.

While our approach could be possibly extended (with additional work and many cases to check) for some small values, it does not seem to be the right way to attack the general problem. In order to stimulate the research, we propose the following weaker variant of the problem, where we ask whether the assumptions for the case $k + 1$ are sufficient to show the statement of the conjecture for k.

Conjecture. Let $k \geq 4$ and let G be a graph with odd-girth$(G) \geq 2k + 3$ and $\text{mad}(G) < \frac{2k+3}{k+1}$. Then G admits a homomorphism to $K(2k + 1, k)$.

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