Analytic Functions*

Piotr Nayar, kolokwium II

Zasady: Trzeba wybrac 5 zadan i zaznaczyc, ze maja one liczyc sie do wyniku z kolokwium. Pozostale zadania tez mozna rozwiazac. Kazde takie zadanie policze jak 3 zadania na pracy domowej. **Mozna korzystac z wlasnych notatek i materialow na moodlu, ale nie mozna szukac informacji w internecie.**

Zadanie 1.

(a) (5p.) Find an explicit holomorphic bijection from $\{\operatorname{Im} z > 0, |z| < 1\}$ onto $\{|z| < 1\}$.

(b) (5p.) Find an explicit holomorphic map from $\{|z| < 1\}$ onto $\{0 < |z| < 1\}$.

Zadanie 2.

(a) (8p.) Let f, g be entire and satisfy $e^f + e^g = 1$. Does this imply that f, g are constant?

(b) (2p.) Let f, g, h be entire and satisfy $e^f + e^g + e^h = 1$. Does this imply that f, g, h are all constant?

Zadanie 3. Suppose (a_n) is a sequence of complex numbers such that $|a_1 + \ldots + a_n| \le 1$ for all $n \ge 1$. Show that $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ defines an analytic function in $\{\operatorname{Re} s > 0\}$.

Zadanie 4. Find all entire functions f having finite order of growth and such that $f(z)f'(z) \neq 0$ for all $z \in \mathbb{C}$.

Zadanie 5. Suppose f is entire and satisfies $f(z + 2\pi i) = f(z)$ for all $z \in \mathbb{C}$. Moreover, let us assume that f has finite order of growth. Show that $f(z) = g(e^z)$ for $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ of the form $g(z) = \sum_{k=-n}^{n} a_k z^k$, where $n \ge 0$ and $a_k \in \mathbb{C}$ for $|k| \le n$.

Zadanie 6. Suppose $P(z)$ is a nonzero polynomial. Show that $e^z + P(z)$ has infinitely many zeros.

Zadanie 7. Prove the identity

$$\frac{\sin \pi z}{z(1-z)} = \prod_{n=1}^{\infty} \left(1 + \frac{z - z^2}{n^2 + n}\right)$$

Zadanie 8. For τ with $\operatorname{Im} \tau > 0$ let $\wp = \wp(z; 1, \tau)$ be the Weierstrass function with periods $1, \tau$.

(a) (3p.) Show that $\wp'' = Q \circ \wp$, where Q is some complex quadratic polynomial.

(b) (7p.) Show that every even meromorphic function with periods $1, \tau$ is of the form $R \circ \wp$, where R is some rational function.